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On Quantum Black Holes ^{*}

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Abstract

A pedagogical discussion is given of some aspects of “quantum black holes”, primarily using recently developed two-dimensional models. After a short preliminary concerning classical black holes, we give several motivations for studying such models, especially the so called dilaton gravity models in $1 + 1$ dimensions. Particularly attractive is the one proposed by Callan, Giddings, Harvey and Strominger (CGHS), which is classically solvable and contains black hole solutions. Its semi-classical as well as classical properties will be reviewed, including how a flux of matter fields produces a black hole with a subsequent emission of Hawking radiation. Breakdown of such an approximation near the horizon, however, calls for exactly solvable variants of this model and some attempts in this direction will then be described. A focus will be placed on a model with 24 matter fields, for which exact quantization can be performed and physical states constructed. A method will then be proposed to extract

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space-time geometry described by these states in the sense of quantum average and examples containing a black hole will be presented. Finally we give a (partial) list of future problems and discuss the nature of difficulties in resolving them.

1 Introduction

Quantum gravity is perhaps the most far-fetched, elusive, enigmatic and dangerous subject in physics. When the fluctuations implied by the word “quantum” is not so large, the danger is not so acutely felt since we are still talking about ripples on some definite space-time. But as the fluctuations become large, we begin to lose virtually all the familiar and useful notions in physics, as the very stage upon which all the events take place, namely the space-time itself, melts away. Worse, we do not even know how to properly quantize gravity in the background independent way. Even if we did and if we could formally solve the theory, it is still hard to interpret the wave function so obtained.

Despite all this, everybody would agree that the subject must contain fascinating and fundamental physics, the prime example of which is that of black holes. Quantum physics of black holes is fascinating because it challenges various basic notions both in quantum mechanics and general relativity. In this lecture, we shall give a pedagogical review of some of the recent developments concerning “quantum black holes”.

But, what is a “quantum black hole”? In fact this nomenclature, which often appears in the literature, does not yet have a definitive meaning. So let us try to give the answer at various levels.

(i) At the least ambitious level, it is used to symbolize various effects due to *quantized matter fields* around a *classical black hole*. Already at this level, fascinating phenomena exist and interesting questions can be asked. To cite a few, the phenomena of Hawking radiation[1], possible loss of information across the event horizon and the related question of the evolution of a pure state into a mixed state[2] (so called the problem of quantum incoherence) can all be recognized at this level, albeit satisfactory answers to these questions would require understanding at higher levels. Also, propagation of strings in a black hole background is an interesting subject in this category.

(ii) At the next level of sophistication, one would like to include at least some of the fluctuations or the deformation of the metric around a black hole, in response to the averaged quantum effects of the matter and/or of the metric itself. This is often called the problem of *back reaction*.

(iii) If one follows the line of classification just described, the final level would be a black hole in *fully quantized* gravity models. One should note a profound gap between the level (ii) and this level. Up to the previous level, the treatment is more or less semi-classical and classically well-defined notion of a black hole is still clearly visible. On the other hand, as will be explicitly illustrated towards the last part of this lecture, physically allowed states in exactly quantizable models will be completely independent of the space-time coordinates as a consequence of the principle of general covariance. Therefore, it becomes quite non-trivial even to identify a black hole. Hard as it may be, it is only at this level that we can hope to answer many of the most intriguing questions concerning black holes. Besides those already mentioned at level (i), these questions include whether the singularity can be washed away by quantum

effects, what the end point of a black hole evaporation would be like, and the statistical mechanical meaning of the entropy of a black hole[3].

(iv) Besides the three levels of increasing degree of sophistication just described, we must mention one more view towards the meaning of a quantum black hole. It is the possibility that in full quantum treatment a black hole might behave like a particle and vice versa[4], [5]. This can be motivated by recalling that given a mass scale M one can form two fundamental length scales, one in quantum mechanics called the Compton wave length L_C and the other in general relativity called the Schwarzschild radius L_{SS} :

$$L_C = \frac{\hbar}{Mc}, \quad L_{SS} = \frac{GM}{c^2}. \quad (1.1)$$

A simple yet important observation is that the Planck mass scale, $M_{Pl} = \sqrt{\hbar c/G}$, is characterized non-perturbatively by that scale at which L_C and L_{SS} are equal. This means that for a black hole with “small” mass, $M < M_{Pl}$, its Schwarzschild radius is *inside* the Compton wave length, just like for elementary particles we know. This may be relevant to the problem of the fate of an evaporating black hole. On the other hand, if what we normally think as a particle has a mass M bigger than M_{Pl} , then its horizon is located *outside* the Compton wave length, a situation typical of a black hole. That such particles naturally appear in string theory makes this observation extremely interesting. In fact, evidences for this phenomenon have been found in recent investigations[6]. This may have a profound bearing on the understanding of the short distance behavior of the string theory.

So, evidently there are many levels at which to talk about “quantum black holes”. In this lecture, we shall discuss aspects (i) \sim (iii) primarily using two-dimensional models, especially those of quantum dilaton gravity.

The content of the rest of the lecture will be as follows: To make it more or less self-contained, we shall begin in Sec.2 with an elementary review of classical black holes.[‡] After recalling the causal structure of flat space-time and the notion of Penrose diagram, we describe the basic properties of a Schwarzschild black hole in several different coordinates. With this preliminary, we then go on in Sec.3 to give several motivations for studying two-dimensional dilaton gravity models, the main subjects of this lecture. One comes from the dimensional reduction of the charged black hole in 3+1 dimensions, especially the so called “extremal solution”, which naturally leads to the dilaton gravity model of Callan, Giddings, Harvey and Strominger (CGHS)[10]. Another is the recent discovery that suitable gauged Wess-Zumino-Witten (WZW) models can be regarded as describing a string in interaction with a black hole in two dimensional target space[11],[12],[13]. Remarks will be made on the difficulties of this approach including that of incorporating the back reaction. In Sec.4, we start our description of dilaton gravity models. CGHS model is introduced and its classical solutions containing a black hole are discussed. Particularly interesting is the solution

[‡]For more detailed description of the content of this section, see for example [7]-[9].

where an incoming matter flux produces a black hole, with a subsequent emission of Hawking radiation. This is analyzed for large number N of matter fields in two different ways. Unfortunately, as the mass of the black hole diminishes, this “semi-classical” approximation fails near the horizon and one is lead to look for exactly solvable models. Examples of such models are described in Sec.5 after discussing an inherent ambiguity in the choice of models. In Sec.6, a variant of CGHS model with 24 massless matter fields is introduced and studied[14] -[16],[17]. This model is shown to be exactly quantizable by a quantum canonical mapping into a set of free fields. Furthermore, by making use of conformal invariance, all the physical states can be constructed in terms of free field oscillators. However, as these states do not depend on space-time coordinates, it is quite non-trivial to understand their physical meaning. In Sec.7, we shall propose a method to extract the space-time geometry they describe in the sense of quantum average[15],[16]. It will be explicitly demonstrated that a black hole geometry emerges by judicious choice of physical states. Finally in Sec.8 we give a (partial) list of future problems and discuss the nature of difficulties in solving them.

The content of the first half of this lecture has a large overlap with that of the excellent reviews [18],[19]. The reader should consult these references as well.

2 Classical Black Holes

2.1 Penrose Diagram

What makes a black hole such a fascinating object is its peculiar causal structure and, as is well-known, it can be displayed in its entirety by a device called Penrose diagram. It is a way of squeezing the whole universe into our hand without messing up the causal order. Let us briefly recall how it works when applied to the flat Minkowski space in $3+1$ dimensions. Adopting the spherical coordinates, the metric is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \quad (2.1)$$

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (2.2)$$

Hereafter we will often suppress the $d\Omega^2$ part. To make the causal structure manifest, define the light-cone variables u and v by

$$u = t - r, \quad v = t + r, \quad (2.3)$$

$$ds^2 = -du dv. \quad (2.4)$$

u and v are of infinite range except for the constraint $v \geq u$, which comes from $v - u = 2r \geq 0$. Now we make a monotonic conformal transformation to define new light-cone variables (v', u') of *finite range*:

$$u = \tan \frac{u'}{2}, \quad v = \tan \frac{v'}{2}, \quad (2.5)$$

$$-\pi \leq u' \leq v' \leq \pi. \quad (2.6)$$

Clearly, the causal property is unchanged under this mapping. From u' and v' one can define new “time” and “space” coordinates t' and r' as

$$t' = \frac{1}{2}(v' + u'), \quad r' = \frac{1}{2}(v' - u'). \quad (2.7)$$

Just like the original radial variable r , $r' \geq 0$ follows from $v' \geq u'$.

The space-time diagram obtained by this procedure is called the Penrose diagram depicted in Fig.1a: The entire space-time is squeezed into a half diamond (with of course a two-sphere attached at each point inside). Besides the preservation of the causal structure, the important feature is that various infinities are clearly distinguished and displayed in the diagram:

i^+ (future time-like infinity) $\iff t \rightarrow \infty$ at finiter,
 i^- (past time-like infinity) $\iff t \rightarrow -\infty$ at finite r ,
 i^0 (space-like infinity) $\iff r \rightarrow \infty$ at finite t ,
 I^+ (future null infinity) $\iff v \rightarrow \infty$ at finite u ,
 I^- (past null infinity) $\iff u \rightarrow -\infty$ at finite v .

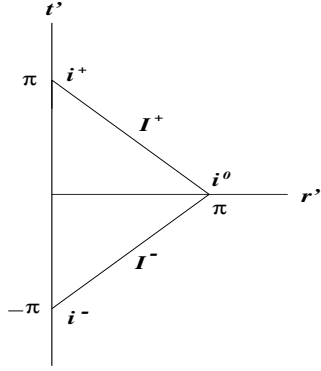


Fig.1a Penrose diagram for flat Minkowski space in 3 + 1 dimensions.

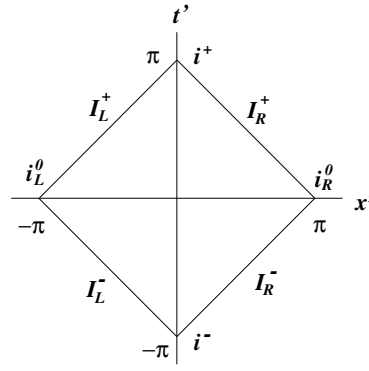


Fig.1b Penrose diagram for flat Minkowski space in 1 + 1 dimensions.

We can follow a similar procedure for 1 + 1 dimensional flat space-time, relevant for our later discussion. In this case, the light-cone variables and the metric are given by

$$x^\pm = t \pm x, \quad -\infty < x < \infty, \quad (2.8)$$

$$ds^2 = -dt^2 + dx^2 = -dx^+ dx^-. \quad (2.9)$$

Because of the range of x , the Penrose diagram, after a conformal transformation, is now over the full diamond as shown in Fig.1b and there are pairs of space-like and null infinities.

2.2 Schwarzschild Black Hole in 3 + 1 dimensions

Let us now apply the same technique to reveal the causal structure of a Schwarzschild black hole. As is customary, we start with its metric in the manifestly asymptotically

flat coordinate system:

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2. \quad (2.10)$$

As r passes through the horizon located at $r = 2M$, the roles of t and r are switched, implying that a light ray can get in but never come out classically. To define a null coordinate system, first factor out $1 - (2M/r)$ and define r^* such that

$$ds^2 = \left(1 - \frac{2M}{r}\right) (-dt^2 + dr^{*2}), \quad (2.11)$$

$$dr^* = \frac{dr}{1 - \frac{2M}{r}}. \quad (2.12)$$

To proceed, we must distinguish the exterior ($r > 2M$) and the interior ($r < 2M$) regions. In the exterior region, the solution of (2.12) is given by

$$r^* = r + 2M \ln \left(\frac{r}{2M} - 1 \right), \quad (2.13)$$

which can also be expressed as

$$e^{r^*/2M} = \frac{r}{2M} \left(1 - \frac{2M}{r}\right) e^{r/2M}. \quad (2.14)$$

r^* varies from $-\infty$ at $r = 2M$ to $+\infty$ at $r = \infty$. The null coordinates, which we shall call u^* and v^* , their ranges, and the metric in terms of them are given by (suppressing $r^2(r^*)d\Omega^2$)

$$u^* = t - r^*, \quad v^* = t + r^*, \quad (2.15)$$

$$-\infty < u^*, v^* < \infty, \quad (2.16)$$

$$ds^2 = - \left(1 - \frac{2M}{r}\right) du^* dv^*. \quad (2.17)$$

Now we wish to absorb the factor in front, which makes the metric singular at the horizon, by performing a conformal transformation to new null coordinates (u, v) , known as the Kruskal coordinates. Noting that this factor occurs in $e^{r^*/2M}$, we define u and v such that

$$du dv \propto e^{r^*/2M} du^* dv^* = e^{(v^* - u^*)/4M} du^* dv^*. \quad (2.18)$$

For u and v to be real, there are two possible pairs of signs:

$$du \propto \pm e^{-u^*/4M} du^*, \quad (2.19)$$

$$dv \propto \pm e^{v^*/4M} dv^*. \quad (2.20)$$

Solving them, we get two regions (I) and (II):

$$\begin{cases} u = -e^{-u^*/4M} \\ v = e^{v^*/4M}, \end{cases} \quad (I) \quad (2.21)$$

$$\begin{cases} u = e^{-u^*/4M} \\ v = -e^{v^*/4M}. \end{cases} \quad (II) \quad (2.22)$$

For both regions the metric takes the same form, which is manifestly non-singular at the horizon:

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} du dv. \quad (2.23)$$

To study the interior region, we must go back to the differential equation (2.12). The appropriate real solution for $r < 2M$ is given by

$$r^* = r + 2M \ln \left(1 - \frac{r}{2M} \right), \quad (2.24)$$

where the range of r^* is $-\infty$ ($r = 2M$) $< r^* \leq 0$ ($r = 0$). Define u^* and v^* exactly as before. Then we get

$$e^{r^*/2M} = e^{(v^*-u^*)/4M} = e^{r/2M} \left(1 - \frac{r}{2M} \right) \quad (2.25)$$

$$= -\frac{2M}{r} e^{r/2M} \left(1 - \frac{2M}{r} \right), \quad (2.26)$$

$$-\infty < u^* \leq v^* < \infty, \quad (2.27)$$

$$u^* = v^* \iff r^* = 0. \quad (2.28)$$

To make the form of ds^2 the same as for the exterior region, we require

$$du dv \propto -e^{(v^*-u^*)/4M} du^* dv^*. \quad (2.29)$$

We then have the following two cases:

$$\begin{cases} u = e^{-u^*/4M} \\ v = e^{v^*/4M}, \end{cases} \quad (III) \quad (2.30)$$

$$\begin{cases} u = -e^{-u^*/4M} \\ v = -e^{v^*/4M}. \end{cases} \quad (IV) \quad (2.31)$$

The constraint $r \geq 0$ translates into $uv \leq 1$. For both regions *III* and *IV*, the line element takes the form

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} du dv, \quad (2.32)$$

which is identical to (2.23). The singularity at $r = 0 \iff uv = 1$ is genuine and is called the black hole singularity.

With the above knowledge, we can draw a u - v diagram called the Kruskal diagram (see Fig.2).

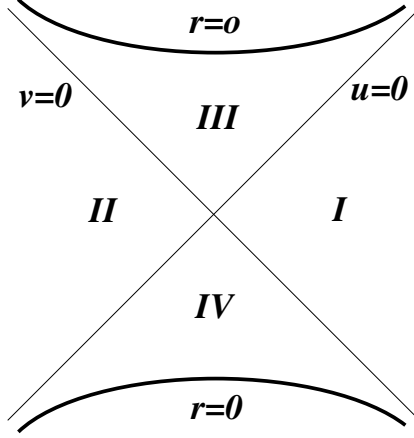


Fig.2 Kruskal diagram for Schwarzschild black hole.

It is clear that the lines $u = 0$ and $v = 0$ separate the fate of the future-directed light rays (*i.e.* whether they can escape to infinity or not) and hence represent the global event horizons.

The Kruskal diagram is still of infinite range. To get the Penrose diagram, we drop the front factor of the metric, which does not change the causal structure, and make the now familiar conformal transformation to new null coordinates (u', v') :

$$u = \tan \frac{u'}{2}, \quad v = \tan \frac{v'}{2}. \quad (2.33)$$

By using the simple identity

$$\tan^{-1} \left(\frac{u+v}{1-uv} \right) = \tan^{-1} u + \tan^{-1} v = \frac{1}{2}(u' + v'), \quad (2.34)$$

we easily see that the singular line at $r = 0$ ($uv = 1$) are mapped into the horizontal straight lines $u' + v' = \pm\pi$. The resultant Penrose diagram is depicted in Fig.3.

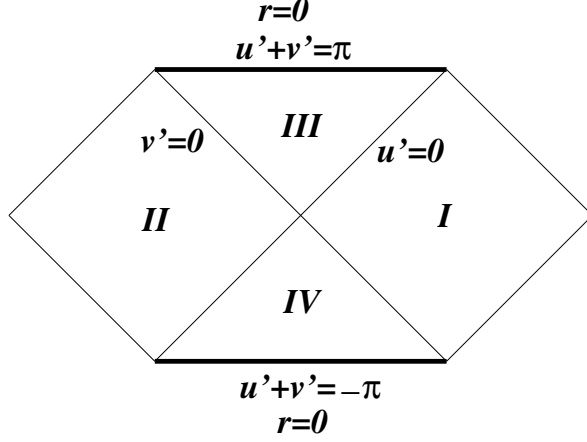


Fig.3 Penrose diagram for Schwarzschild black hole.

For later comparison, it is of interest to look at the form of the singularity in the Kruskal coordinates. By expressing the factor $(1/r)e^{-r/2M}$ in terms of u and v , we find the square root type behavior

$$ds^2 \simeq -8\sqrt{2}M^2 \frac{1}{\sqrt{1-uv}} du dv. \quad (2.35)$$

Note that the strength of the singularity is not expressed in the Penrose diagram.

3 Motivation for 2 Dimensional Dilaton Gravity Models

3.1 Black Hole in 3 + 1 Dimensional Dilaton Gravity and Dimensional Reduction

One of the characteristic features of (super)string theory is that a scalar field called dilaton is always associated with the graviton and plays an important role in gravitational physics. As it is massless (at least perturbatively), it remains in the low energy effective action after suitable compactification down to 3 + 1 dimensions. The simplest among such actions contains, in addition, a $U(1)$ gauge field and takes the following form:

$$S = \int dx^4 \sqrt{-g} e^{-2\phi} \left(R + 4g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} \right). \quad (3.1)$$

Although this is the natural form following from string theory (for example by β function analysis of the corresponding non-linear sigma model), it apparently has

two “unusual” features. First, with the factor $e^{-2\phi}$, which plays the role of the string coupling constant squared g_s^2 , the Einstein term does not have the canonical form. Furthermore, the kinetic term for the dilaton has the “wrong” sign. (We are using the space-favored signature convention.) These features are not necessarily problematic since there is always an (admittedly annoying) ambiguity in the choice of the conformal frame. In fact, if we make a conformal transformation of the form

$$\tilde{g}_{\mu\nu} = e^{-2\phi} g_{\mu\nu}, \quad (3.2)$$

both of the above “problems” are removed simultaneously and we get the canonical form:

$$S = \int d^4x \sqrt{-\tilde{g}} \left(\tilde{R} - 2\tilde{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} e^{-2\phi} \tilde{g}^{\mu\lambda} \tilde{g}^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} \right). \quad (3.3)$$

We shall follow the usual terminology and call $\tilde{g}_{\mu\nu}$ the “canonical” metric and $g_{\mu\nu}$ the “string metric”. In this model, there is no potential for the dilaton field. This means that one can choose any constant value ϕ_0 for it at infinity, which therefore is a parameter of the model. This parameter has a definite physical meaning in the canonical frame: The exponential factor $e^{2\phi_0}$, which appears in front of the gauge field kinetic term, can be identified as the electromagnetic coupling squared at ∞ .

Although this model has a variety of black hole solutions[20], [21], we shall focus on the simplest non-rotating one with a magnetic charge Q . Its line element in the canonical frame is given by

$$\begin{aligned} d\tilde{s}^2 &= \tilde{\gamma}_{\mu\nu} dx^\mu dx^\nu \\ &= - \left(1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{(1 - 2M/r)} \\ &\quad + r^2 \left(1 - \frac{Q^2}{2Mr} e^{-2\phi_0} \right) d\Omega^2, \end{aligned} \quad (3.4)$$

$$e^{-2\phi} = e^{-2\phi_0} \left(1 - \frac{Q^2}{2Mr} e^{-2\phi_0} \right), \quad (3.5)$$

$$F = Q \sin \theta d\theta \wedge d\phi. \quad (3.6)$$

Notice that the gauge field configuration is precisely that of an abelian monopole. The metric is quite similar to that of Schwarzschild solution except that the area of the spatial two-sphere is proportional to the factor $\left(1 - \frac{Q^2}{2Mr} e^{-2\phi_0} \right)$, which also appears in $e^{-2\phi}$. One can check that a curvature singularity develops as this factor vanishes. It is also easy to see that the horizon is located at $r = 2M$ just like in the Schwarzschild case.

What will be relevant to 1 + 1 dimensional dilaton gravity model is a special case of the above solution called the “extremal solution”. It occurs at a specific value of the magnetic charge, namely at $Q^2 = 4M^2 e^{2\phi_0}$, where the factor for $e^{-2\phi}$ becomes

exactly the one specifying the horizon:

$$e^{-2\phi} = e^{-2\phi_0} \left(1 - \frac{2M}{r}\right). \quad (3.7)$$

As seen in the canonical frame, this is a peculiar limit where the singularity and the horizon coincide and at the same time the area of the two-sphere vanishes. In the string metric, however, the situation looks quite different. As we have to multiply by $e^{2\phi}$ to go to the string metric, the singularity in the angular part and in front of dt^2 disappear simultaneously and we get

$$ds^2 = e^{2\phi_0} \left(-dt^2 + \frac{dr^2}{(1 - (2M/r))^2} + r^2 d\Omega^2 \right). \quad (3.8)$$

The asymptotic behavior for large r is unchanged and we still have a flat space. To investigate what happens near $r = 2M$ let us introduce the Kruskal coordinate σ (previously denoted by r^*):

$$d\sigma = \frac{dr}{1 - 2M/r}, \quad (3.9)$$

$$e^{\sigma/2M} = \frac{r}{2M} e^{r/2M} \left(1 - \frac{r}{2M}\right). \quad (3.10)$$

The metric and the dilaton field then take the forms

$$ds^2 = e^{-2\phi_0} \left(-dt^2 + d\sigma^2 + r^2(\sigma) d\Omega^2 \right), \quad (3.11)$$

$$e^{-2\phi} = e^{-2\phi_0} e^{\sigma/2M} \frac{2M}{r} e^{-r/2M}. \quad (3.12)$$

Now it is easy to see what happens near $r = 2M$. We get

$$ds^2 \longrightarrow e^{2\phi_0} \left(-dt^2 + d\sigma^2 + (2M)^2 d\Omega^2 \right), \quad (3.13)$$

$$\phi \longrightarrow \phi_0 - \frac{\sigma}{2M}. \quad (3.14)$$

Eq.(3.13) says that we have a two-sphere of fixed radius $2M$ and the space-time is flat. As for the dilaton, it becomes linear in σ , where $\sigma \rightarrow -\infty$ as seen from (3.12). This configuration is often referred to as the *linear dilaton vacuum*. Since the two-sphere is effectively frozen, this suggests that near $r = 2M$ the essence of the dynamics should be describable by a two-dimensional theory. Employing the technique of dimensional reduction, one obtains a simple action:

$$S = \frac{1}{\gamma^2} \int d\xi^2 \sqrt{-g} e^{-2\phi} \left(R + 4(\nabla\phi)^2 + 4\lambda^2 - \frac{1}{2}F^2 \right), \quad (3.15)$$

$$\lambda^2 = \frac{1}{4Q^2}. \quad (3.16)$$

Furthermore, if there are no charged particles, we can ignore the two-dimensional gauge fields since there will be no local dynamics. Thus we arrive at a simple model of two-dimensional dilaton gravity proposed by Callan, Giddings, Harvey and Strominger[10]. We shall describe its classical and quantum properties in detail in Sec.4.

3.2 $SL(2)/U(1)$ Black Hole in 1 + 1 Dimensions

Another motivation for studying 1+1 dimensional theories comes from the discovery[11], [12], [13] that a suitable gauged WZW model, the prototype of which is based on the coset $SL(2, R)/U(1)$, can be regarded as an exactly conformally invariant non-linear sigma model describing a string theory in a black hole background. As it is the first explicit model capable of describing the interaction of a string with a black hole, vigorous investigations have been performed. However, as we shall discuss in a moment, this approach has a couple of serious shortcomings, including the inherent inability to take into account the back reaction of the metric. In this respect, the field theoretic model introduced at the end of the previous subsection would be more versatile. For this reason, we shall keep the exposition very brief.

Let us begin with the description of the ungauged WZW model based on $SL(2, R)$. Its action is given by

$$S(g) = L(g) + i\Gamma(g), \quad (3.17)$$

$$L(g) = \frac{k}{8\pi} \int_{\Sigma} \sqrt{h} h^{ij} \text{Tr} \left(g^{-1} \partial_i g g^{-1} \partial_j g \right), \quad (3.18)$$

$$\Gamma(g) = \frac{k}{12\pi} \int_B \text{Tr} \left(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \right), \quad (3.19)$$

where k is so called the level, which plays the role of the inverse coupling constant. The model has a global symmetry group $SL(2, R)_L \otimes SL(2, R)_R$ acting on $g \in SL(2, R)$ as AgB^{-1} , with $A \in SL(2, R)_L$ and $B \in SL(2, R)_R$.

As the group is non-compact, there is a unitarity problem as it stands. However, this can be cured if one can gauge away the unwanted states. One way of achieving this is to gauge the anomaly-free $U(1)$ subgroup generated by σ_3 . Explicitly,

$$A = B = 1 + \epsilon \sigma_3 + \cdots, \quad (3.20)$$

$$\delta g = \epsilon \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g + g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \quad (3.21)$$

where ϵ is an infinitesimal gauge function. A convenient parametrization for g is

$$g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix}, \quad ab + uv = 1. \quad (3.22)$$

After introducing the $U(1)$ gauge field A_μ such that $\delta A_\mu = -\partial_\mu \epsilon$, one must fix a gauge. From the form of the $U(1)$ generator, it is clear that demanding a relation

between a and b does the job. The simplest choice which respects the constraint $ab + uv = 1$ is to take $a = b$ for $1 - uv > 0$ and $a = -b$ for $1 - uv < 0$. The condition for conformal invariance reads $c = (3k/(k - 2)) - 1 = 26$, where -1 is due to the gauging of $U(1)$. Hence the level should be taken to be $k = 9/4$. Upon performing the integration over A_μ , the action takes the form of a non-linear sigma model

$$L = -\frac{k}{4\pi} \int d^2x \sqrt{-h} \frac{h^{\mu\nu} \partial_\mu u \partial_\nu v}{1 - uv}, \quad (3.23)$$

from this one can immediately read off the target space line element:

$$ds^2 = -\frac{du dv}{1 - uv}. \quad (3.24)$$

This evidently describes a black hole in a Kruskal-like coordinate. (As we shall see, this form is essentially identical to the black hole solution in the dilaton gravity model of CGHS.) Actually, proper treatment of the integration measure for A_μ generates a target space dilaton field coupled to the two-dimensional curvature scalar R . One can also introduce the tachyon field and study the Hawking radiation[22].

Now we would like to make some remarks on the serious shortcomings of the sigma model formulation of string theory in curved background in general.

(i) The first unsatisfactory feature is the apparent redundancy in the description of the metric degrees of freedom. While the fluctuations of the metric corresponding to gravitons are described as excitations of a string, the macroscopic part of the metric appears explicitly in the action and is regarded as describing the space-time in which the very string propagates. Since coherent states of gravitons should be able to represent the macroscopic space-time as well, there must at least be some consistency condition between the two. In other words, the correct formulation must be invariant under the choice of decomposition of the background and the fluctuation.

(ii) We may illuminate the same kind of problem in a different way. Normally, to incorporate the fluctuations of a background field one tries to integrate over the relevant field, in this case the target space metric $G_{\mu\nu}$. But since only for a special class of $G_{\mu\nu}$ is the model conformally invariant, integration over all possible metric will inevitably require some off-shell formulation of string theory, which is outside the scope of the sigma model description. String field theory would be the prime candidate but its present day formulation does not appear to be suitable for black hole physics. Another possibility is to insist on the conformal invariance and integrate only over the moduli space with some measure. The problem here is that there does not appear to be any principle of determining the appropriate measure.

(iii) It should by now be clear that the gist of the problem is the lack of background independent formulation of string theory. Manifestation of this can be seen already at the level of low energy effective field theory: If one can identify the moduli parameters as expectation values of some fields, one would have a background independent formulation. However, the potential for these fields will be flat and one cannot determine the preferred direction.

With the above discussion, it is fair to say that at present there is no background independent formulation of string theory in which one can discuss quantum black holes. One can only describe a black hole as a rigid background geometry and hence cannot address the question of back reaction. (A possible exception may be the attempts to describe black hole geometry in suitable matrix models[23]. It is interesting to see if one can overcome the difficult problem of space-time interpretation in that approach.)

4 Semi-Classical Black Holes in 1 + 1 Dimensional Dilaton Gravity Models

With the motivations just described, we shall now begin our discussion of two-dimensional dilaton gravity models of CGHS type.

4.1 CGHS Model

The action of the original CGHS model is given by[10]

$$S = \frac{1}{\gamma^2} \int d^2\xi \sqrt{-g} \left\{ e^{-2\phi} [R + 4(\nabla\phi)^2 + 4\lambda^2] - \frac{1}{2} \sum_{i=1}^N (\nabla f_i)^2 \right\}, \quad (4.1)$$

where f_i ($i = 1, 2, \dots, N$) are massless matter scalar fields. Let us take the conformal gauge with flat background and use the light-cone variables. Relevant expressions are

$$g_{\mu\nu} = e^{2\rho} \eta_{\mu\nu}, \quad (4.2)$$

$$\xi^\pm = \xi^0 \pm \xi^1, \quad \square = \eta^{\mu\nu} \partial_\mu \partial_\nu = -4\partial_+ \partial_-, \quad (4.3)$$

$$g_{+-} = -\frac{1}{2} e^{2\rho}, \quad g^{+-} = -2e^{-2\rho}, \quad (4.4)$$

$$R_{\mu\nu} = -\eta_{\mu\nu} \square \rho, \quad R_{+-} = -2\partial_+ \partial_- \rho, \quad (4.5)$$

$$R = -2e^{-2\rho} \square \rho = 8e^{-2\rho} \partial_+ \partial_- \rho. \quad (4.6)$$

Then the action becomes

$$S = \frac{4}{\gamma^2} \int d^2\xi \left\{ e^{-2\phi} [2\partial_+ \partial_- \rho - 4\partial_+ \phi \partial_- \phi + \lambda^2 e^{2\rho}] + \frac{1}{2} \sum \partial_+ f_i \partial_- f_i \right\}. \quad (4.7)$$

As far as the classical analysis is concerned, it is made extremely simple by making a change of variable by a conformal transformation to the “canonical metric” $\tilde{g}_{\mu\nu}$ discussed in the previous section. Defining

$$\Phi \equiv e^{-2\phi}, \quad (4.8)$$

the transformation is expressed as

$$g_{\mu\nu} = \Phi^{-1} \tilde{g}_{\mu\nu}, \quad (4.9)$$

$$\sqrt{-g} = \Phi^{-1} \sqrt{-\tilde{g}}, \quad (4.10)$$

$$R = \Phi \left(\tilde{R} - 2\tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi \right). \quad (4.11)$$

Notice that in two-dimensions the transformation factor for $\sqrt{-g}$ in (4.10) is Φ^{-1} and not Φ^{-2} as in four-dimensions. Because of this, the term involving the scalar curvature will not be of Einstein form (which is a total derivative in two-dimensions). Also, from (4.11) one immediately sees that the kinetic terms for ϕ cancel. The resultant action is

$$S = \frac{1}{\gamma^2} \int d^2\xi \sqrt{-\tilde{g}} \left(\Phi \tilde{R} + 4\lambda^2 - \frac{1}{2} \sum (\tilde{\nabla} f_i)^2 \right). \quad (4.12)$$

Now we choose a conformal gauge with the conformal factor denoted by ψ :

$$\tilde{g}_{\mu\nu} = e^\psi \eta_{\mu\nu}, \quad (4.13)$$

$$S = \frac{4}{\gamma^2} \int d^2\xi \left(-\frac{1}{2} (\partial_+ \Phi \partial_- \psi + \partial_+ \psi \partial_- \Phi) + \lambda^2 e^\psi + \frac{1}{2} \sum \partial_+ f_i \partial_- f_i \right). \quad (4.14)$$

The equations of motion following from the variations $\delta\psi$, $\delta\Phi$ and δf_i are quite simple:

$$\delta\psi : \quad \partial_+ \partial_- \Phi + \lambda^2 e^\psi = 0, \quad (4.15)$$

$$\delta\Phi : \quad \partial_+ \partial_- \psi = 0, \quad (4.16)$$

$$\delta f_i : \quad \partial_+ \partial_- f_i = 0. \quad (4.17)$$

The second equation tells us that ψ is a free field and can be decomposed into the right- and left-going parts:

$$\psi = \psi_+(\xi^+) + \psi_-(\xi^-). \quad (4.18)$$

To solve the first equation (4.15), define two functions $A(\xi^+)$ and $B(\xi^-)$ such that

$$\partial_+ A = \lambda e^{\psi_+}, \quad (4.19)$$

$$\partial_- B = \lambda e^{\psi_-}. \quad (4.20)$$

Then (4.15) takes the form

$$\partial_+ \partial_- (\Phi + AB) = 0, \quad (4.21)$$

showing that the combination $\Phi + AB$ is again a free field, which we call χ . Thus we have

$$\Phi = -(\chi + AB), \quad (4.22)$$

$$\chi = \chi_+(\xi^+) + \chi_-(\xi^-). \quad (4.23)$$

In addition to the equations of motion, we must impose the vanishing of the energy-momentum tensor, which follows from general covariance. First, T_{+-} , which is proportional to the trace T^μ_μ for flat background we are considering, is given by

$$T_{+-} = -(\partial_+ \partial_- \Phi + \lambda^2 e^\psi), \quad (4.24)$$

where we have set $\gamma^2 = 4\pi$ for convenience. This automatically vanishes from the equation of motion (4.15) showing that the system has conformal invariance. The remaining components $T_{\pm\pm}$ take the form

$$T_{\pm\pm} = -\partial_\pm \Phi \partial_\pm \psi + \partial_\pm^2 \Phi + \frac{1}{2} \sum (\partial_\pm f_i)^2 \quad (4.25)$$

$$= \partial_\pm \chi \partial_\pm \psi - \partial_\pm^2 \chi + T_{\pm\pm}^f, \quad (4.26)$$

where the expression in terms of the free fields ψ and χ (the second line) is obtained by the substitution $\Phi = -(\chi + AB)$. To solve $T_{++} = 0$, let us note the identity

$$\partial_+ (\partial_+ \chi e^{-\psi_+}) = -(\partial_+ \chi \partial_+ \psi - \partial_+^2 \chi) e^{-\psi_+}. \quad (4.27)$$

Thus $T_{++} = 0$ reads $\partial_+ (\partial_+ \chi e^{-\psi_+}) = e^{-\psi_+} T_{++}^f$ and it can be integrated to yield

$$\chi_+ = \text{const.} + \int^{\xi^+} du e^{\psi_+(u)} \int^u dv e^{-\psi_+(v)} T_{++}^f(v). \quad (4.28)$$

Solution for χ_- is entirely similar.

Before we go on, it would be useful to summarize what we got so far: (For simplicity we shall hereafter use λ and $1/\lambda$ as our mass and length units respectively and set $\lambda = 1$.)

$$g^{\mu\nu} = e^{-2\rho} \eta^{\mu\nu} = \Phi e^{-\psi} \eta^{\mu\nu}, \quad (4.29)$$

$$\psi = 2(\rho - \phi) = \psi_+(\xi^+) + \psi_-(\xi^-) = \text{free field}, \quad (4.30)$$

$$\Phi = e^{-2\phi} = -(\chi + AB), \quad (4.31)$$

$$\chi = \chi_+(\xi^+) + \chi_-(\xi^-) = \text{free field}, \quad (4.32)$$

$$\partial_+ A = e^{\psi_+(\xi^+)}, \quad \partial_- B = e^{\psi_-(\xi^-)}, \quad (4.33)$$

$$T_{\pm\pm} = \partial_\pm \chi \partial_\pm \psi - \partial_\pm^2 \chi + \frac{1}{2} \sum (\partial_\pm f_i)^2, \quad (4.34)$$

$$\chi_+ = \text{const.} + \int^{\xi^+} du e^{\psi_+(u)} \int^u dv e^{-\psi_+(v)} T_{++}^f(v). \quad (4.35)$$

Now let us describe some interesting classical solutions of this system. Analysis is the simplest in the gauge (within the conformal gauge) where $\psi = 0$ *i.e.* $\rho = \phi$, which we shall call the Kruskal gauge. The reason why we can impose such a condition is because ρ , and hence ψ , transforms inhomogeneously under conformal transformation.

First, let us consider the case where matter fields are absent *i.e.* $f_i = 0$ for all i and hence $T_{\pm\pm}^f = 0$. Taking into account the gauge condition, we readily obtain from (4.33) and (4.35)

$$\chi = \text{const.} \equiv -M, \quad (4.36)$$

$$A = \xi^+, \quad B = \xi^- \quad (\text{up to a constant shift}). \quad (4.37)$$

Therefore, ρ , ϕ , the line element and the curvature scalar become

$$e^{-2\rho} = e^{-2\phi} = M - \xi^+\xi^-, \quad (4.38)$$

$$ds^2 = -\frac{d\xi^+d\xi^-}{M - \xi^+\xi^-}, \quad (4.39)$$

$$R = 8e^{-2\rho}\partial_+\partial_-\rho = \frac{4M}{M - \xi^+\xi^-}. \quad (4.40)$$

This is similar to the Schwarzschild solution in the Kruskal coordinate and in fact exactly the same as the $SL(2)/U(1)$ black hole, with M as the mass of the black hole. In Fig.4 we show the Penrose diagram for this configuration.

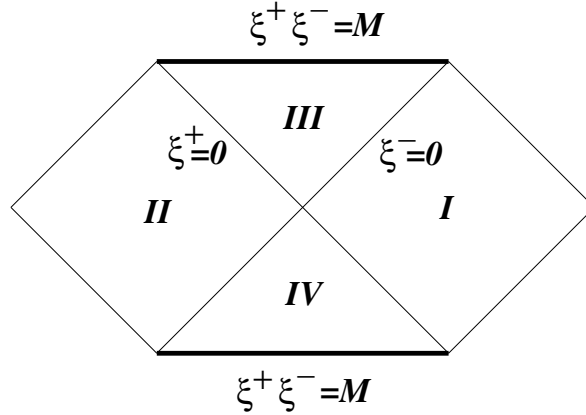


Fig.4 Penrose diagram for a black hole in CGHS model.

Note the horizons at $\xi^+ = 0$ and $\xi^- = 0$ and the curvature singularity along $\xi^+\xi^- = M$.

From the expression of R we see that as $|\xi^+\xi^-| \rightarrow \infty$ the space-time becomes flat. The coordinates in which this becomes manifest can easily be constructed by a conformal transformation. In region I ($\xi^+ > 0, \xi^- < 0$) the transformation, which is similar to the one between the Kruskal and (u^*, v^*) coordinates in the Schwarzschild case, is given by

$$\xi^+ = e^{\sigma^+}, \quad d\xi^+ = \xi^+ d\sigma^+, \quad (4.41)$$

$$\xi^- = -e^{-\sigma^-}, \quad d\xi^- = -\xi^- d\sigma^-. \quad (4.42)$$

The line element then becomes

$$\begin{aligned}
ds^2 &= -e^{2\rho(\sigma)} d\sigma^+ d\sigma^- \\
&= -\frac{d\sigma^+ d\sigma^-}{1 + Me^{-(\sigma^+ - \sigma^-)}} \\
&\xrightarrow{\sigma \rightarrow \infty} -d\sigma^+ d\sigma^-,
\end{aligned} \tag{4.43}$$

where σ is the spatial coordinate defined by $2\sigma \equiv \sigma^+ - \sigma^-$. Notice that ρ , not being a scalar, got non-trivially transformed. In contrast, ϕ is a genuine scalar and we still have $e^{-2\phi} = M + \xi^+ \xi^- = M + e^{\sigma^+ - \sigma^-}$. In the asymptotically flat region it behaves like

$$\begin{aligned}
\phi(\sigma) &= -\sigma - \frac{1}{2} \ln(1 + Me^{-2\sigma}) \\
&\xrightarrow{\sigma \rightarrow \infty} -\sigma,
\end{aligned} \tag{4.44}$$

and we have a linear dilaton vacuum.

Let us recall that $e^{2\phi}$ carries the meaning of g_s^2 , the string coupling constant squared. It represents the strength of the joining-splitting interaction of strings in the original theory and in the present context indicates the strength of the gravitational coupling. From the expression above it is easy to see that (reinstating the scale λ)

$$\begin{aligned}
g_s^2 &\sim \frac{\lambda}{M} && \text{near the horizon,} \\
&\sim 0 && \text{in the linear dilaton region.}
\end{aligned} \tag{4.45}$$

We see that near the horizon the coupling becomes large as M becomes small.

So much for the case without matter fields. Now we shall discuss a more interesting solution which describes a formation of a black hole by a flux of matter fields. For definiteness, suppose a pulse of left-going matter flux is sent in between ξ_i^+ and ξ_f^+ . It is represented by the matter energy-momentum tensor of the form

$$T_{++}^f(v) = F(v)(\theta(v - \xi_i^+) - \theta(v - \xi_f^+)), \tag{4.46}$$

where $F(v)$ is a positive function specifying the profile of the flux (see Fig.5).

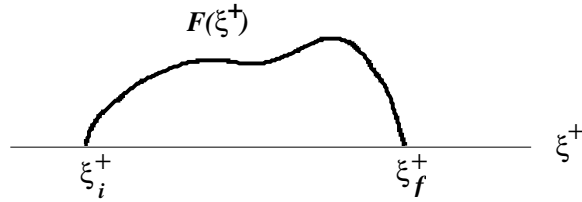


Fig.5 Profile of left-going matter flux.

Assume we have a linear dilaton vacuum before the matter is sent in. Then, from the general formula (4.35), χ is given by

$$\begin{aligned}\chi &= \theta(\xi^+ - \xi_i^+) \int_{\xi_i^+}^{\xi^+} du \int_{\xi_i^+}^u dv T_{++}^f(v) \\ &= \begin{cases} 0 & \xi^+ \leq \xi_i^+ \\ H(\xi^+) & \xi_i^+ \leq \xi^+ \leq \xi_f^+ \\ (\xi^+ - \xi_f^+)a + b & \xi_f^+ \leq \xi^+ \end{cases},\end{aligned}\quad (4.47)$$

where

$$H(u) \equiv \int_{\xi_i^+}^u dw \int_{\xi_i^+}^w dv F(v), \quad (4.48)$$

$$H'(u) = \int_{\xi_i^+}^u dv F(v), \quad (4.49)$$

$$b = H(\xi_f^+), \quad a = H'(\xi_f^+). \quad (4.50)$$

Thus *after the matter has collapsed in*, the metric is of the form

$$\begin{aligned}e^{-2\rho} &= e^{-2\phi} = -(\chi + AB) \\ &= (\xi_f^+ a - b) - \xi^+(\xi^- + a) \\ &= M - \xi^+(\xi^- + a).\end{aligned}\quad (4.51)$$

M is defined as $M \equiv \xi_f^+ a - b$ and it can be shown to be positive for any positive $F(v)$.

The behavior of the scalar curvature can be readily computed. It is given by

$$R = \begin{cases} 0 & \xi^+ \leq \xi_i^+ \\ -\frac{4(\xi^+ H'(\xi^+) - H(\xi^+))}{H(\xi^+) + \xi^+ \xi^-} & \xi_i^+ \leq \xi^+ \leq \xi_f^+ \\ \frac{4M}{M - \xi^+(\xi^- + a)} & \xi_f^+ \leq \xi^+ \end{cases}. \quad (4.52)$$

Before the matter flux is sent in, R naturally vanishes. After the flux has past, it takes exactly the same form as for the previously discussed black hole solution without matter, except with an important shift $\xi^- \rightarrow \xi^- + a$ caused by the flux. The behavior in the transient region is a smooth interpolation of these two. The Penrose diagram of this space-time is shown in Fig.6.

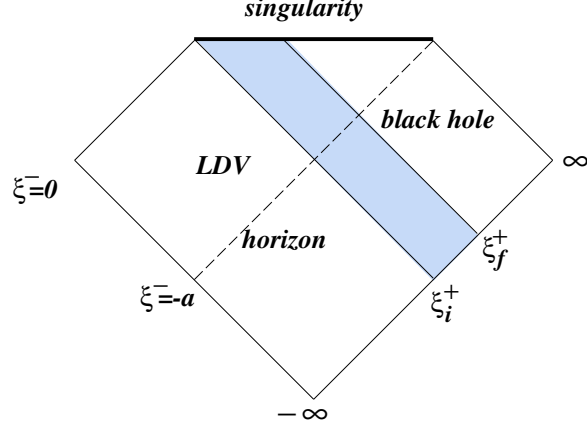


Fig.6 Penrose diagram for a black hole generated by a matter flux. (LDV stands for linear dilaton vacuum.)

A particularly simple and useful choice of the flux is the so called “shock wave” configuration described by the δ -function

$$F(v) = a\delta(\xi^+ - \xi_0^+), \quad (4.53)$$

$$M = \xi_f^+ a - b = a\xi_0^+. \quad (4.54)$$

Clearly the transient region is compressed into a line and many simplifications occur.

In the space-time with a black hole generated by a matter flux, there are two distinct asymptotically flat regions and hence two corresponding (asymptotically) flat coordinates. In the “in” region, where we have the linear dilaton vacuum, we introduce (ζ^+, ζ^-) by

$$\xi^+ = e^{\zeta^+}, \quad \xi^- = -e^{-\zeta^-}, \quad (4.55)$$

$$ds^2 = -d\zeta^+ d\zeta^-, \quad (4.56)$$

while for the “out” region along the future right null infinity I_R^+ , the proper coordinates (σ^+, σ^-) are

$$\xi^+ = e^{\sigma^+}, \quad \xi^- = -e^{-\sigma^-} - a, \quad (4.57)$$

$$ds^2 = -\frac{d\sigma^+ d\sigma^-}{1 + Me^{-(\sigma^+ - \sigma^-)}}. \quad (4.58)$$

These two coordinate systems are related by the conformal transformation

$$\begin{cases} \zeta^+ = \sigma^+ \\ \zeta^- = -\ln(e^{-\sigma^-} + a) \end{cases}. \quad (4.59)$$

This will be important in discussing the Hawking radiation.

It is instructive to express the parameter M in terms of the energy-momentum tensor as seen by these asymptotic observers. Using the classical transformation property of T_{++}^f and integration by parts, the quantities a and b take the form

$$a = \int d\xi^+ T_{++}^f(\xi^+) = \int d\sigma^+ e^{-2\sigma^+} T_{++}^f(\sigma^+), \quad (4.60)$$

$$\begin{aligned} b &= \int du e^u \int^u d\sigma^+ e^{-\sigma^+} T_{++}^f(\sigma^+) \\ &= e^{\sigma_f^+} a - \int_{\sigma_i^+}^{\sigma_f^+} T_{++}^f(\sigma^+). \end{aligned} \quad (4.61)$$

Therefore M is expressed as

$$M = \xi_f^+ a - b = \int_{\sigma_i^+}^{\sigma_f^+} T_{++}^f(\sigma^+). \quad (4.62)$$

We see that M is precisely the total matter energy pumped in and, since the energy conservation holds in the asymptotically flat region, it appears as the mass of the black hole.

4.2 Large N Analysis of Hawking Radiation

Having seen that classically a black hole is generated by a matter flux, we now start incorporating quantum effects upon this configuration.

The first question is whether the famous Hawking radiation exists in this model. One way of analyzing this problem is to make use of the remarkable relation between the trace anomaly and Hawking radiation valid in two dimensions first spelled out by Christensen and Fulling[24]. In classically conformally invariant theory in two dimensions, the expectation value of the trace $\langle T_\mu^\mu \rangle$ can only be proportional to R since no other local scalar of dimension two is available. Indeed by direct calculation one can show that for models with N massless bosonic fields, one has

$$\langle T_\mu^\mu \rangle = \frac{N}{24} R. \quad (4.63)$$

In the light-cone coordinates, $\langle T_\mu^\mu \rangle = 2g^{+-} \langle T_{+-} \rangle = -4e^{-2\rho} \langle T_{+-} \rangle$ and $R = 8e^{-2\rho} \partial_+ \partial_- \rho$ so that the above relation is equivalent to

$$\langle T_{+-} \rangle = -\frac{N}{12} \partial_+ \partial_- \rho. \quad (4.64)$$

To show its relevance to the Hawking radiation, Christensen and Fulling first solve $\nabla_\mu T_\nu^\mu = 0$ in the reduced two-dimensional Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2, \quad (4.65)$$

assuming T_μ^ν is t independent. The resultant T_μ^ν has, in general, non-zero trace. Then they compare it with the general form of T_μ^ν for massless radiation and find that the anomaly is precisely due to such radiation.

Let us apply this procedure for the CGHS model[10]. The covariant conservation equation reads

$$\begin{aligned} 0 &= \nabla^\mu T_{\mu\nu} = g^{\mu\alpha} \nabla_\alpha T_{\mu\nu} \\ &= g^{\mu\alpha} \left(\partial_\alpha T_{\mu\nu} - \Gamma_{\alpha\mu}^\beta T_{\beta\nu} - \Gamma_{\alpha\nu}^\beta T_{\mu\beta} \right). \end{aligned} \quad (4.66)$$

Set $\nu = +$ and evaluate it in conformal gauge with $\langle T_{+-} \rangle = -(N/12)\partial_+\partial_-\rho$. We then get a differential equation for $\langle T_{++} \rangle$ (the one for T_{--} is entirely similar)

$$\begin{aligned} \partial_- \langle T_{++} \rangle &= 2\partial_+\rho \langle T_{+-} \rangle - \partial_+ \langle T_{-+} \rangle \\ &= -\frac{N}{12}\partial_- \left((\partial_+\rho)^2 - \partial_+^2 \rho \right). \end{aligned} \quad (4.67)$$

It is immediately integrated to yield

$$T_{\pm\pm} = -\frac{N}{12} \left((\partial_\pm \rho)^2 - \partial_\pm^2 \rho + t_\pm \right), \quad (4.68)$$

where t_\pm are functions which should be determined from appropriate boundary conditions in the asymptotically flat region. A natural boundary condition is that $\langle T_{\pm\pm} \rangle = 0$ in the linear dilaton region. Using the form $e^{-2\rho} = 1 + ae^{-\sigma^-}$, this yields

$$\begin{cases} t_+ = 0, \\ t_- = - \left((\partial_- \rho)^2 - \partial_-^2 \rho \right) = -\frac{1}{4} \left(1 - \frac{1}{(1+ae^{-\sigma^-})^2} \right). \end{cases} \quad (4.69)$$

With these t_\pm it is easy to check that at the right null infinity I_R^+ (*i.e.* as $\sigma^+ \rightarrow \infty$) the components $\langle T_{++} \rangle$ and $\langle T_{+-} \rangle$ vanish, while the left-going flux is given by

$$\langle T_{--} \rangle \xrightarrow{\sigma^+ \rightarrow \infty} \frac{N}{48} \left(1 - \frac{1}{(1+ae^{-\sigma^-})^2} \right). \quad (4.70)$$

This can be interpreted as the Hawking radiation[10].

The derivation just presented is elegant but somewhat obscure since the meaning of the expectation value has not been clearly defined. A more transparent calculation along the line of the original derivation of Hawking is instructive[25]. The strategy is to first define the out-vacuum $|0\rangle_{out}$ as seen from the observer at I_R^+ to be the state in which ${}_{out}\langle 0|T|0\rangle_{out} = 0$ holds, where T stands for T_{--} . This implicitly determines the normal ordering of T . Actually, however, what is realized is the in-vacuum $|0\rangle_{in}$ which does not contain any quanta defined in the linear dilaton region. Since $|0\rangle_{in}$

and $|0\rangle_{out}$ are related by a Bogoliubov transformation, what the out-observer sees, namely ${}_{in}\langle 0|T|0\rangle_{in}$, can be non-trivial.

The actual calculation can be set up as follows. Let us denote by $x(=\sigma^-)$ and $y(=\zeta^-)$ the coordinate for out and in regions respectively. The mode expansion for the matter field $f(y)$ and the definition of the in-vacuum are given by

$$f(y) = \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} (a_\omega e^{-i\omega y} + a^\dagger e^{i\omega y}) , \quad (4.71)$$

$$[a_\omega, a_{\omega'}^\dagger] = \delta(\omega - \omega') , \quad (4.72)$$

$$a_\omega |0\rangle_{in} = 0 . \quad (4.73)$$

Now we want to evaluate the expectation value ${}_{in}\langle 0|T|0\rangle_{in}$ as seen in the out region. Rather than performing the normal ordering explicitly, it is more convenient to regularize the operator product appearing in T by the point-splitting method and later subtract out ${}_{out}\langle 0|T|0\rangle_{out}$. Thus we make a conformal transformation $y = y(x)$ and define $y_\delta \equiv y(x + \delta)$ with δ infinitesimal. Then T in the out-region is given by

$$T(x) = \frac{1}{2} \partial_x f(y(x)) \partial_x f(y(x + \delta)) . \quad (4.74)$$

By a simple calculation, the expectation value in question is computed as

$$\begin{aligned} {}_{in}\langle 0|T(x)|0\rangle_{in} &= \frac{1}{2} {}_{in}\langle 0|\partial_x f(y) \partial_x f(y_\delta)|0\rangle_{in} \\ &= \frac{1}{4} y' y'_\delta \int_0^\infty d\omega \omega e^{i\omega(y_\delta - y + i\epsilon)} \\ &= -\frac{1}{4} \frac{y' y'_\delta}{(y_\delta - y)^2} , \end{aligned} \quad (4.75)$$

where the prime means $\partial/\partial x$. We now expand this expression in powers of δ . If we define

$$R_1 \equiv \frac{y''}{y'} , \quad R_2 \equiv \frac{y'''}{y'} , \quad (4.76)$$

we get the expansion

$$y'_\delta = y' \left(1 + \delta R_1 + \frac{1}{2} \delta^2 R_2 + \cdots \right) , \quad (4.77)$$

$$(y_\delta - y)^{-2} \simeq \delta^{-2} y'^{-2} \left(1 - \delta R_1 + \delta^2 \left(\frac{3}{4} R_1^2 - \frac{1}{3} R_2 \right) \right) , \quad (4.78)$$

$$\begin{aligned} -\frac{1}{4} \frac{y' y'_\delta}{(y_\delta - y)^2} &\simeq -\frac{1}{4\delta^2} - \frac{1}{24} \left(R_2 - \frac{3}{2} R_1^2 \right) \\ &\simeq -\frac{1}{4\delta^2} - \frac{1}{24} \{y(x), x\} , \end{aligned} \quad (4.79)$$

where

$$\{y(x), x\} = \partial_x^2 \ln y' - \frac{1}{2}(\partial_x \ln y')^2 \quad (4.80)$$

is the Schwarzian derivative. One should note that this calculation is identical to that of the conformal anomaly for the Virasoro operator. As for ${}_{out}\langle 0|T|0\rangle_{out}$, the Schwarzian derivative term is absent and we simply get

$${}_{out}\langle 0|T|0\rangle_{out} = -\frac{1}{4\delta^2}. \quad (4.81)$$

As stated before the normal ordering with respect to $|0\rangle_{out}$ is effected by subtracting this divergence. Thus for N matter fields, we finally get

$$\begin{aligned} {}_{in}\langle 0|T_{--}^{f,reg}(x)|0\rangle_{in} &= -\frac{N}{24}\{y(x), x\} \\ &= -\frac{N}{24}\left((\ln \zeta^{-'})'' - \frac{1}{2}(\ln \zeta^{-'})^2\right) \\ &= \frac{N}{48}\left(1 - \frac{1}{(1 + ae^{-\sigma^-})^2}\right), \end{aligned} \quad (4.82)$$

which gives exactly the same Hawking radiation as obtained previously.

4.3 Effect of Back Reaction and Difficulties

Although it is gratifying that the model is capable of describing Hawking radiation, the expression we have got cannot be the correct one: It gives a divergent answer when integrated over the future right null infinity. The cause of this problem is obvious. As the black hole emits radiation its mass must diminish, but this effect has not been taken into account. A possible way to tackle this back reaction problem is to incorporate some quantum effects in the form of an effective action and re-solve the equations of motion and energy-momentum constraints.

The simplest of such approximation schemes is to consider the limit of large N while keeping $Ne^{2\phi}$ finite. This has two merits: (i) Large N corresponds to small $e^{2\phi}$, hence weak coupling, and one-loop contribution should dominate. (ii) The only one-loop contribution of order N is the anomaly due to the matter fields, and one can avoid the quantization of the dilaton-gravity sector which is rather difficult. Thus, with the anomaly term of $\mathcal{O}(N)$, the effective action becomes

$$\begin{aligned} S &= \frac{4}{\gamma^2} \int d^2\xi \left\{ e^{-2\phi} [2\partial_+ \partial_- \rho - 4\partial_+ \phi \partial_- \phi + \lambda^2 e^{2\rho}] \right. \\ &\quad \left. - \frac{N}{12} \partial_+ \rho \partial_- \rho + \frac{1}{2} \sum \partial_+ f_i \partial_- f_i \right\}. \end{aligned} \quad (4.83)$$

It turns out that the equations of motion following from this action can no longer be solved in closed form. Moreover, the system becomes singular in a certain region. A way to see this is to look at the kinetic matrix (“target space metric”) and its determinant:

$$K = - \begin{pmatrix} 4e^{-2\phi} & 2e^{-2\phi} \\ 2e^{-2\phi} & \frac{N}{12} \end{pmatrix}, \quad (4.84)$$

$$\det K = 4e^{-2\phi} \left(\frac{N}{12} - e^{-2\phi} \right). \quad (4.85)$$

Evidently the determinant vanishes for $e^{-2\phi} = \frac{N}{12}$, which occurs within the weak coupling region.

Dispite these difficulties, various attempts have been made to extract some physical consequences from this model[26]- [38]. For instance, it has been shown that an apparent horizon (*i.e.* boundary of trapped points) forms outside the event horizon and approaches the latter as time goes on. This can be regarded as an indication of the reduction of the mass of the black hole. Another information is that, at least in this model, the classical black hole singularity is not resolved by the large N quantum effects. The most discouraging feature of this approximation is that it is bound to fail in the most interesting regime near the end point of the evaporation. In that region M becomes very small compared with λ and hence the coupling constant $g_s^2 \sim \lambda/M$ becomes large, as was already mentioned in (4.45).

5 Solvable Models of Dilaton Gravity

We have seen that although the CGHS model appears to have many attractive features it is hard to go beyond the large N (weak coupling) approximation. In such an approximation dilaton-gravity degrees of freedom are not quantized and furthermore one cannot satisfactorily discuss what happens in the the strong coupling region near the horizon, especially for small M . Thus, it is natural to seek some variants of the CGHS model which are fully quantizable. In this section, we will discuss a strategy for finding them and describe an application of such an idea.

5.1 Freedom in the Choice of the Model

In searching for a good candidate, one must bear in mind that there is an immense freedom in the possible form of the action for dilaton gravity models in two dimensions: Since the dilaton field is dimensionless it is easy to construct an infinite number of models which are power-counting renormalizable. Thus, including the counter terms, the form of the action is highly ambiguous. Also as long as the general covariance is respected, one can choose a variety of functional measures. With this in mind, a large class of models can be represented as a non-linear sigma model, familiar in

string theory, of the form [39], [40]

$$S = -\frac{1}{\gamma^2} \int d^2\xi \sqrt{-\hat{g}} \left(\frac{1}{2} G_{\mu\nu}(X) \hat{g}^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu + \Phi(X) \hat{R} + T(X) \right), \quad (5.1)$$

where X^λ are coordinates in the $\rho - \phi$ space and, in the string theory language, $G_{\mu\nu}(X)$, $\Phi(X)$ and $T(X)$ are, respectively, the target space metric, the (target space) dilaton and the tachyon fields.

Of course not every sigma model can be interpreted as a model of gravity in two dimensions: It must possess two dimensional general covariance. In the conformal gauge $g_{\mu\nu} = e^{2\rho} \hat{g}_{\mu\nu}$, this requirement means that the action should be independent of the separation of the conformal factor and the background. More specifically, the action should be invariant under the transformation

$$\hat{g} \longrightarrow e^{2\delta\omega} \hat{g}, \quad (5.2)$$

$$\rho \longrightarrow \rho - \delta\omega, \quad (5.3)$$

$$\phi \longrightarrow \phi. \quad (5.4)$$

In the string theory context, the consequence of this requirement is well-known: The β functions for the (coupling) fields $G_{\mu\nu}$, Φ and T must vanish. Up to one loop, these equations take the form

$$\beta_{\mu\nu}^G = 2\nabla_\mu \nabla_\nu \Phi + \mathcal{R}_{\mu\nu} - \nabla_\mu T \nabla_\nu T + \dots = 0, \quad (5.5)$$

$$\begin{aligned} \beta^\Phi &= 4(\nabla\Phi)^2 - 4\nabla^2\Phi - \mathcal{R} + \frac{N-24}{3} \\ &\quad + (\nabla T)^2 - 2T^2 + \dots = 0, \end{aligned} \quad (5.6)$$

$$\beta^T = 4\nabla_\mu \Phi \nabla^\mu T - 4T - 2\nabla^2 T + \dots = 0, \quad (5.7)$$

where the covariant derivatives and the curvatures are those in the target space.

5.2 A Solvable Models for $N \neq 24$

Let us give an example of solvable models of this type proposed by de Alwis[39] and independently by Bilal and Callan[41]. As the sigma model aspect is clearer, we follow [39]. First assume that for weak coupling $e^{2\phi} \ll 1$ the model reduces to that of CGHS with an addition of an anomaly term, namely to

$$\begin{aligned} S &= \frac{1}{\gamma} \int d^2\xi \sqrt{-\hat{g}} \left\{ e^{-2\phi} \left[\hat{R} + 4(\hat{\nabla}\phi)^2 - 4\hat{\nabla}\phi \cdot \hat{\nabla}\rho + 4\lambda^2 e^{2\rho} \right] \right. \\ &\quad \left. + \kappa \left((\hat{\nabla}\rho)^2 + \hat{R}\rho \right) \right\}. \end{aligned} \quad (5.8)$$

In terms of the sigma model functions, this means

$$\begin{aligned} G_{\phi\phi} &= -8e^{-2\phi}, \quad G_{\phi\rho} = 4e^{-2\phi}, \quad G_{\rho\rho} = -2\kappa, \\ \Phi &= -e^{-2\phi} - \kappa\rho, \quad T = -4\lambda^2 e^{2(\rho-\phi)}. \end{aligned} \quad (5.9)$$

Now introduce functions $h(\phi)$, $\bar{h}(\phi)$ and $\bar{\bar{h}}(\phi)$ of order $e^{2\phi}$ which express deviations of $G_{\mu\nu}$ from the CGHS form and write the target space line element as

$$\begin{aligned} ds^2 = & -8e^{-2\phi} [1 + h(\phi)] d\phi^2 \\ & + 8e^{-2\phi} [1 + \bar{h}(\phi)] d\rho d\phi \\ & - 2\kappa(1 + \bar{\bar{h}}(\phi)) d\rho^2. \end{aligned} \quad (5.10)$$

It can be shown that for $\bar{\bar{h}} = 0$ the target space curvature \mathcal{R} vanishes. Restricting to that case, there must exist a target space coordinate system in which the metric becomes manifestly flat. It turns out that the following transformation $(\rho, \phi) \rightarrow (x, y)$ does the job:

$$y \equiv \rho + \frac{1}{\kappa} e^{-2\phi} - \frac{2}{\kappa} \int d\phi e^{-2\phi} \bar{h}(\phi), \quad (5.11)$$

$$x \equiv \int d\phi e^{-2\phi} \left((1 + \bar{h})^2 - \kappa e^{2\phi} (1 + h) \right)^{1/2}, \quad (5.12)$$

$$ds^2 = \frac{8}{\kappa} dx^2 - 2\kappa dy^2. \quad (5.13)$$

Next, de Alwis analyzes the β function equations with small T approximation. The solution so obtained can be summarized in the following form of the action (including the massless matter fields f_i)

$$\begin{aligned} S = & \frac{1}{\gamma^2} \int d^2\xi \left\{ \mp (\partial_+ X \partial_- X - \partial_+ Y \partial_- Y) + \sum_i \partial_+ f_i \partial_- f_i \right. \\ & \left. + 2\lambda^2 e^{\mp \sqrt{2/|\kappa|} (X \mp Y)} \right\}, \end{aligned} \quad (5.14)$$

where

$$\kappa = \frac{N - 24}{6}, \quad (5.15)$$

$$X \equiv 2\sqrt{2/|\kappa|} x, \quad Y \equiv \sqrt{2|\kappa|} y. \quad (5.16)$$

It can be shown that by a further transformation this action can be mapped on to a conformally invariant system described in terms of free fields. Such a system is obviously classically solvable (with the inclusion of the anomaly term). Furthermore it is apparently solvable even fully quantum mechanically. However, there is a serious problem: Since the transformation $(\rho, \phi) \longleftrightarrow (X, Y)$ shown above is highly non-linear and non-local, the relation between the original and the transformed operators as well as the functional measure taken are quite obscure. For this reason, essentially classical analysis has been performed for these models.

Note added: After this talk was delivered, we have found that a large class of solvable models (including those discussed above and the one which will be treated

in detail in the next two sections) can be characterized as non-linear sigma models *with a special symmetry*. (This symmetry is an extensive generalization of the one discussed in [42].) With that observation, the discussions presented in this section can be made much more transparent and general. Interested reader should be referred to [43].

6 An Exactly Quantizable Model of CGHS Type for $N = 24$

As we have seen, there are enormous possibilities for models of dilaton gravity in two dimensions, even within solvable models. In this section, we shall discuss one particular model, for which exact quantization can be carried out and all the physical states can be constructed [14]. An attempt to extract the averaged geometry they describe, including that of a black hole, will be presented in the next section.

6.1 Choice of the Model

In order to define the quantum model we deal with, first recall the classical CGHS model in the “canonical” metric $\tilde{g}_{\mu\nu}$ discussed before:

$$S = \frac{1}{\gamma^2} \int d^2\xi \sqrt{-\tilde{g}} \left(\Phi \tilde{R} + 4\lambda^2 - \frac{1}{2} \sum (\tilde{\nabla} f_i)^2 \right), \quad (6.1)$$

$$\Phi \equiv e^{-2\phi}, \quad g_{\mu\nu} = \Phi^{-1} \tilde{g}_{\mu\nu}. \quad (6.2)$$

As was remarked at that time, the kinetic term for Φ is absent in this expression. We now reinstate it by a conformal transformation of the form

$$\tilde{g}_{\mu\nu} = e^\Phi h_{\mu\nu}, \quad (6.3)$$

$$\sqrt{-\tilde{g}} = e^\Phi \sqrt{-h}, \quad (6.4)$$

$$\tilde{R} = e^{-\Phi} (R_h - h_{\mu\nu} \nabla_\mu \nabla_\nu \Phi). \quad (6.5)$$

This results in a Liouville like action, first written down by Russo and Tseytlin [44]:

$$S = \frac{1}{\gamma^2} \int d^2\xi \sqrt{-h} \left((\nabla \Phi)^2 + 4\lambda^2 e^\Phi + \Phi R_h - \frac{1}{2} \sum (\nabla f_i)^2 \right). \quad (6.6)$$

To quantize this model, we must specify the functional measure. One possible choice is the measure induced by the functional norms of the form [45]

$$\begin{aligned} \|\delta h\|^2 &= \int d^2x \sqrt{-h} h^{\alpha\beta} h^{\gamma\delta} \delta h_{\alpha\gamma} h_{\beta\delta}, \\ \|\delta \Phi\|^2 &= \int d^2x \sqrt{-h} \delta \Phi \delta \Phi, \\ \|\delta f_i\|^2 &= \int d^2x \sqrt{-h} \delta f_i \delta f_i \quad (i = 1, \dots, N), \end{aligned} \quad (6.7)$$

which are natural with respect to the metric $h_{\mu\nu}$. Now adopt the conformal gauge

$$h_{\alpha\beta} = e^{2\rho_h} \hat{g}_{\alpha\beta}, \quad (6.8)$$

and perform the famous David-Distler-Kawai analysis [46] to change to the standard translationally invariant measure. The result is the effective action

$$\begin{aligned} S = & \frac{1}{\gamma^2} \int d^2\xi \sqrt{-\hat{g}} \left[\nabla_h \Phi \hat{\nabla}_h \Phi + 2 \nabla_h \Phi \hat{\nabla}_h \rho_h + \hat{R} \Phi \right. \\ & + 4\lambda^2 e^{\Phi+2\rho_h} + \gamma^2 \frac{N-24}{24\pi} (\nabla_h \rho_h \hat{\nabla}_h \rho_h + \hat{R} \rho_h) \\ & \left. - \frac{1}{2} \nabla_h \vec{f} \hat{\nabla}_h \vec{f} \right] + S^{gh}(\hat{g}, b, c), \end{aligned} \quad (6.9)$$

where hatted quantities are computed with the reference metric $\hat{g}_{\mu\nu}$ and S^{gh} is the usual ghost action. Although everything is clear and straightforward, the procedure above is not without problems. One, which is frequently encountered in other dilaton gravity models as well, is that the positivity of Φ is not properly respected. Another problem is that it is not clear in which conformal frame we should interpret the model physically. If we choose to interpret the physics in the original frame, namely with respect to $g_{\mu\nu}$, the classical black hole singularity occurs at $\Phi = 0$ and fluctuations in the region of negative Φ , the first problem, might not be so relevant. In any case, as we do not have satisfactory answers to these problems, we shall proceed with the model and watch if anything pathological will happen. Furthermore, we will focus on the special case with $N = 24$ in which the term due to the anomaly drops out and the model simplifies considerably. Essentially the same model was analyzed from a different point of view in [17].

To perform a rigorous quantization, we will put the system in a “box” (*i.e.* an interval) of size L and impose the periodic boundary conditions. This is best implemented by introducing the dimensionless variables

$$x^\alpha = (t, \sigma) = \xi^\alpha / L, \quad \mu = \lambda L, \quad (6.10)$$

and demand that all the fields appearing in the action be 2π -periodic in σ .

As far as the classical analysis is concerned, it is identical to that for the CGHS model. For convenience, let us display the relevant results obtained before (see Eqs.(4.29) \sim (4.35)), with the notational replacements $\rho \longrightarrow \rho_g$, $\rho_h \longrightarrow \rho$.

$$g^{\mu\nu} = e^{-2\rho_g} \eta^{\mu\nu} = \Phi e^{-\psi} \eta^{\mu\nu} = \Phi e^{-(\Phi+2\rho)} \eta^{\mu\nu}, \quad (6.11)$$

$$\psi = \psi_+(x^+) + \psi_-(x^-) = \text{free field}, \quad (6.12)$$

$$\rho = \frac{1}{2}(\psi - \Phi), \quad (6.13)$$

$$\Phi = e^{-2\phi} = -(\chi + AB), \quad (6.14)$$

$$\chi = \chi_+(\xi^+) + \chi_-(\xi^-) = \text{free field}, \quad (6.15)$$

$$\partial_+ A = \mu e^{\psi_+(x^+)}, \quad \partial_- B = \mu e^{\psi_-(x^-)}, \quad (6.16)$$

$$\tilde{\gamma}^2 T_{\pm\pm} = \partial_{\pm} \chi \partial_{\pm} \psi - \partial_{\pm}^2 \chi + \frac{1}{2} \sum (\partial_{\pm} f_i)^2, \quad (6.17)$$

$$\tilde{\gamma} \equiv \frac{\gamma}{\sqrt{4\pi}}. \quad (6.18)$$

Previously, the solutions for $A(x^+)$ and $B(x^-)$ were considered only in the $\psi = 0$ gauge. As we wish to utilize the full conformal invariance, we do not want to impose that restriction here. Moreover, we need to solve for these functions with proper boundary conditions. In fact this problem is exactly the same as the one which occurs in the operator treatment of the Liouville theory [47], [48]. To find the boundary conditions for A and B , we write out the mode expansion for ψ . For the left-moving part, it can be written as

$$\psi^+ = \tilde{\gamma} \left\{ \frac{q^+}{2} + p^+ x^+ + i \sum_{n \neq 0} \frac{\alpha_n^+}{n} e^{-inx^+} \right\}. \quad (6.19)$$

and similarly for ψ^- . Under $\sigma \rightarrow \sigma + 2\pi$, ψ^{\pm} undergoes a shift which depends on the *zero mode* p^+ . This means that although the product AB is periodic A and B separately are not and they undergo the change

$$A(x^+ + 2\pi) = \alpha A(x^+), \quad (6.20)$$

$$B(x^- - 2\pi) = \frac{1}{\alpha} B(x^-), \quad (6.21)$$

where $\alpha = e^{\gamma\sqrt{\pi}p^+}$. The solutions satisfying these boundary conditions take the form (suppressing the t dependence)

$$A(\sigma) = \mu C(\alpha) \int_0^{2\pi} d\sigma' E_{\alpha}(\sigma - \sigma') e^{\psi_+(\sigma')}, \quad (6.22)$$

$$B(\sigma) = \mu C(\alpha) \int_0^{2\pi} d\sigma'' E_{1/\alpha}(\sigma - \sigma'') e^{\psi_-(\sigma'')}. \quad (6.23)$$

Here $C(\alpha) = 1/(\sqrt{\alpha} - \sqrt{\alpha}^{-1})$ and the function $E_{\alpha}(\sigma)$ is a variant of a step function, shown in Fig.7, given by

$$E_{\alpha}(\sigma) \equiv \exp\left(\frac{1}{2} \ln \alpha \epsilon(\sigma)\right). \quad (6.24)$$

$\epsilon(\sigma)$ is the usual stair step function with the property $\epsilon(\sigma + 2\pi) = 2 + \epsilon(\sigma)$.

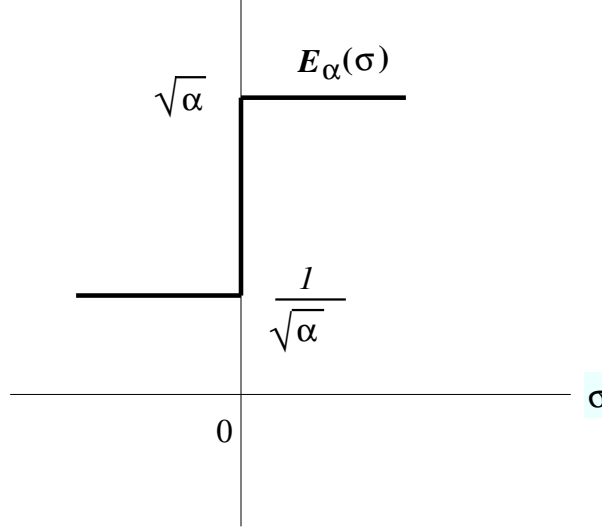


Fig.7 Sketch of the step function $E_\alpha(\sigma)$.

Let us now go back to the energy-momentum tensor $T_{\pm\pm}$ given in (6.17). Introduce ϕ_1, ϕ_2 and ϕ_f^i by

$$\psi = \frac{\tilde{\gamma}}{\sqrt{2}}(\phi_1 + \phi_2), \quad \chi = \frac{\tilde{\gamma}}{\sqrt{2}}(\phi_1 - \phi_2), \quad (6.25)$$

$$f^i = \tilde{\gamma}\phi_f^i. \quad (6.26)$$

Then $T_{\pm\pm}$ become diagonal:

$$T_{\pm\pm} = \frac{1}{2}(\partial_\pm \phi_1)^2 - Q\partial_\pm^2 \phi_1 - \frac{1}{2}(\partial_\pm \phi_2)^2 + Q\partial_\pm^2 \phi_2 + \frac{1}{2}(\partial_\pm \vec{\phi}_f)^2, \quad (6.27)$$

where the background charge Q is given by

$$Q = \frac{\sqrt{2\pi}}{\gamma}. \quad (6.28)$$

Note ϕ_2 is of *negative metric*.

It is not difficult to obtain the general solutions of $T_{\pm\pm} = 0$ satisfying the proper boundary conditions. To display them, it is convenient to write $A(x^+)$ and $B(x^-)$ in the form

$$A(x^+) = \mu e^{\tilde{\gamma} p^+ x^+} a(x^+), \quad (6.29)$$

$$B(x^-) = \mu e^{\tilde{\gamma} p^+ x^-} b(x^-), \quad (6.30)$$

where $a(x^+)$ and $b(x^-)$ are arbitrary periodic functions. Then the general solutions are spanned by $a(x^+), b(x^-)$ and χ and ψ , $T_{\pm\pm}^f$ and $g_{\alpha\beta}$ are expressed in terms of them. In particular, the original metric takes the form

$$\begin{aligned} g_{\alpha\beta} &= \frac{e^\psi}{\chi + AB} \eta_{\alpha\beta} \\ &= \frac{e^\psi}{\chi + \mu^2 e^{2\tilde{\gamma} p^+ t} a(x^+) b(x^-)} \eta_{\alpha\beta}. \end{aligned} \quad (6.31)$$

Let us give an example. A configuration which describes a matter-free black hole in the large L limit can be produced by the choice

$$\chi = -c = \text{constant}, \quad \partial_\pm \vec{f} = 0, \quad (6.32)$$

$$a(x^+) = \sin x^+, \quad b(x^-) = \sin x^-, \quad (6.33)$$

for which ψ becomes

$$\psi_\pm = \tilde{\gamma} p^+ x^\pm + \ln(\tilde{\gamma} p^+ \sin x^\pm + \cos x^\pm). \quad (6.34)$$

Recalling the definitions $x^\alpha = \xi^\alpha/L$ and $\mu = \lambda L$, we indeed get

$$\begin{aligned} \lim_{L \rightarrow \infty} g_{\alpha\beta} &= \lim_{L \rightarrow \infty} \frac{-e^\psi}{c - \mu^2 e^{2\tilde{\gamma} p^+ t} \sin x^+ \sin x^-} \eta_{\alpha\beta} \\ &= -\frac{1}{c - \lambda^2 \xi^+ \xi^-} \eta_{\alpha\beta}, \end{aligned} \quad (6.35)$$

which describes a simple black hole. A black hole produced by a matter shock wave can similarly be constructed.

6.2 Quantization of the Model

Since the original fields are expressed in terms of free fields, we expect that the usual quantization of the latter achieves the full quantization of the model. We can confirm this by showing that the non-linear and non-local transformation $(\Phi, \rho) \rightarrow (\psi, \chi)$ is a canonical transformation quantum mechanically as well as classically.

Let us write the mode expansion for the free bosons ϕ_i , ($i = 1, 2$) as

$$\phi_i(x^+, x^-) = \phi_i^+(x^+) + \phi_i^-(x^-), \quad (6.36)$$

$$\phi_i^\pm(x^\pm) = \frac{q^i}{2} + p^i x^\pm + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^{(i, \pm)} e^{-in x^\pm}, \quad (6.37)$$

and assume the standard free field Poisson bracket relations :

$$i \{ \alpha_m^{(1, \pm)}, \alpha_n^{(1, \pm)} \} = m \delta_{m+n, 0}, \quad \{ q^1, p^1 \} = 1, \quad (6.38)$$

$$i \{ \alpha_m^{(2, \pm)}, \alpha_n^{(2, \pm)} \} = -m \delta_{m+n, 0}, \quad \{ q^2, p^2 \} = -1, \quad (6.39)$$

$$\text{Rest} = 0. \quad (6.40)$$

One can then easily compute the general Poisson brackets between the fields ψ and χ . The result shows that, as can be guessed from the form of $T_{\pm\pm}$, ψ and χ are conjugate to each other and $\{\psi(x), \psi(y)\} = \{\chi(x), \chi(y)\} = 0$. Now express the original fields (Φ, ρ) and their conjugate momenta $(\Pi_\Phi, \Pi_\rho) = (2(\dot{\Phi} + \dot{\rho}), 2\dot{\Phi})$ in terms of (ψ, χ) and compute the equal time (ET) Poisson brackets among them. Calculations are tedious due to the presence of non-local expressions and zero mode factors, but they lead to the expected result

$$\{\Phi(x), \Pi_\Phi(y)\}_{ET} = \delta(\sigma_x - \sigma_y), \quad (6.41)$$

$$\{\rho(x), \Pi_\rho(y)\}_{ET} = \delta(\sigma_x - \sigma_y), \quad (6.42)$$

$$\text{Rest} = 0, \quad (6.43)$$

showing that the transformation is canonical at least classically.

To quantize the system, we intend to make the usual replacement $\{\phi_i, \phi_j\}_{ET} \rightarrow \frac{\hbar}{i} [\phi_i, \phi_j]$. When the transformation is non-linear, its canonical nature can be destroyed in this process due to the normal-ordering necessary for the composite operators involved. Fortunately for the present model this does not happen: The dangerous composite operator AB consists only of ψ field and its components commute among themselves as we remarked earlier. Thus *normal ordering is unnecessary* and the proof of canonicity goes through exactly as in the classical case. In this respect, this model is far simpler than the bona fide Liouville theory.

After the quantization the model continues to enjoy conformal invariance without anomaly. The Virasoro generators L_n^{dL} and L_n^f , for the dilaton-Liouville (dL) and the matter (f) sector respectively, are given by

$$L_n^{dL} = L_n^1 + L_n^2, \quad (6.44)$$

$$L_n^1 = +\frac{1}{2} \sum_m : \alpha_{n-m}^1 \alpha_m^1 : + iQn\alpha_n^1, \quad (6.45)$$

$$L_n^2 = -\frac{1}{2} \sum_m : \alpha_{n-m}^2 \alpha_m^2 : - iQn\alpha_n^2, \quad (6.46)$$

$$L_n^f = \frac{1}{2} \sum_m : \vec{\alpha}_{n-m}^f \cdot \vec{\alpha}_m^f :, \quad (6.47)$$

where Q is as given in (6.28). Note the overall negative sign for L_n^2 due to the negative metric associated with the oscillators α_n^2 . When we compute the algebra of these operators, the Q -dependence cancel out and we find the usual form given by

$$[L_m^{dL}, L_n^{dL}] = (m-n)L_{m+n}^{dL} + \frac{2}{12}(m^3 - m)\delta_{m+n,0}, \quad (6.48)$$

$$[L_m^f, L_n^f] = (m-n)L_{m+n}^f + \frac{N}{12}(m^3 - m)\delta_{m+n,0}. \quad (6.49)$$

Evidently for $N = 24$ the central charges add up to the critical value $c = 26$. With respect to the total generator L_n^{tot} , the free fields ψ and χ transform as

$$[L_m^{tot}, \psi(x)] = e^{imx^+} \left(\frac{1}{i} \partial_+ \psi + im \right), \quad (6.50)$$

$$\left[L_m^{tot}, \chi(x) \right] = e^{imx^+} \frac{1}{i} \partial_+ \chi. \quad (6.51)$$

Thus, while χ is a genuine conformal field with dimension 0, ψ transforms with an additional inhomogeneous term. Because of this contribution, $\dim e^{\lambda\psi} = \lambda$ and in particular the operator $e^{\lambda\psi}$ appearing in the action is a $(1,1)$ operator. As for $e^{\lambda\chi}$, its dimension is 0 regardless of λ . Another operator of importance is AB . By a somewhat tedious calculation, one can show that, despite its complexity, it is a dimension 0 primary:

$$[L_n, AB(x)] = e^{inx^+} \frac{1}{i} \frac{\partial}{\partial x^+} (AB(x)). \quad (6.52)$$

This is desirable since we want $\Phi = -(\chi + AB)$ to have a definite transformation property.

6.3 BRST Analysis of Physical States

Having quantized the model exactly, the next task is to construct the physical states[14]. As the reader must have noticed, the mathematical structure of the model is identical to that of a bosonic string theory in 26 dimensions *with a special background charge*. Thus techniques developed for string theory can be readily transcribed to this case with minor modifications. Specifically, we shall apply the BRST method which was quite successful in analyzing the physical states of string models with central charge $c \leq 1$ [49][50]. (Analysis similar to the one below was also performed in [51][52].)

We shall explicitly deal only with the left-moving sector and use chiral fields $\phi_1(x^+)$ and $\phi_2(x^+)$ previously introduced. The mode expansion and the basic commutation relations are given by

$$\phi_i(x^+) = \frac{1}{2} q_i + p_i x^+ + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^i e^{-inx^+}, \quad (6.53)$$

$$\partial_+ \phi_i = \sum_{n \in \mathbf{Z}} \alpha_n^i e^{-inx^+} \quad (\alpha_0^i \equiv p^i), \quad (6.54)$$

$$[q_1, p_1] = i\hbar, \quad [q_2, p_2] = -i\hbar, \quad (6.55)$$

$$[q_1, p_2] = [q_2, p_1] = 0, \quad (6.56)$$

$$[\alpha_m^1, \alpha_n^1] = -[\alpha_m^2, \alpha_n^2] = m\hbar \delta_{m+n,0}, \quad [\alpha_m^1, \alpha_n^2] = 0. \quad (6.57)$$

In the standard fashion we can construct the BRST operator d as

$$d = \sum c_{-n} (L_n^{dL} + L^f n) - \frac{1}{2} \sum : (m-n) c_{-m} c_{-n} b_{m+n} :, \quad (6.58)$$

where the normal ordering for the ghost and the anti-ghost oscillators, c_n and b_n , is with respect to the so-called “down vacuum” $|0\rangle_{\downarrow}$, which is related to the SL_2 -invariant vacuum by $|0\rangle_{\downarrow} = c_1 |0\rangle_{inv}$.

In the following, our emphasis will be on the general strategy for finding physical states and therefore we often omit technical details. In the BRST formalism, a physical state is characterized as a state which is annihilated by d (BRST-closed) yet cannot be written as $d | \Lambda \rangle$ for some state $| \Lambda \rangle$ (BRST-non-exact). Thus, a physical state $| \Psi \rangle$ can always be written in the form

$$| \Psi \rangle = | \Psi_0 \rangle + d | \Lambda \rangle, \quad (6.59)$$

where $| \Psi_0 \rangle$, which cannot be of the form $d | * \rangle$, will be referred to as the non-trivial part. Mathematically, states satisfying these conditions are said to form d -cohomology classes or *absolute cohomology* classes. Since d is a rather complicated operator it is hard to find these states directly. The basic idea is to reduce the problem to that of finding simpler cohomologies embedded in the original and with that information reconstruct the absolute cohomology.

The first step is to decompose the operator d with respect to the ghost zero modes c_0 and b_0 :

$$d = c_0 L_0^{tot} - M b_0 + \hat{d}, \quad (6.60)$$

where L_0^{tot} is the total Virasoro generator at level 0, M is a complicated operator the structure of which does not concern us, and \hat{d} does not contain c_0 nor b_0 . From the above decomposition it is easy to see that L_0^{tot} can be written as an anti-commutator

$$L_0^{tot} = \{b_0, d\}. \quad (6.61)$$

This is important since it implies that the non-trivial part $| \Psi_0 \rangle$ of a physical state must satisfy $L_0^{tot} | \Psi_0 \rangle = 0$. The proof is elementary: Suppose the global weight of $| \Psi_0 \rangle$ is h , i.e., $L_0^{tot} | \Psi_0 \rangle = h | \Psi_0 \rangle$. We can rewrite this as

$$\begin{aligned} L_0^{tot} | \Psi_0 \rangle &= h | \Psi_0 \rangle \\ &= (db_0 + b_0 d) | \Psi_0 \rangle = d(b_0 | \Psi_0 \rangle). \end{aligned} \quad (6.62)$$

Hence if $h \neq 0$ we can divide by h and find that $| \Psi_0 \rangle = d \left(\frac{1}{h} b_0 | \Psi_0 \rangle \right)$, showing that it is not a physical state. As this contradicts the assumption, we must have $h = 0$.

With this in mind, we can now define a subspace of states called the space of *relative cohomology* \mathcal{F}_0 as follows:

$$\mathcal{F}_0 \equiv \left\{ | \psi \rangle \mid L_0^{tot} | \psi \rangle = 0, b_0 | \psi \rangle = 0 \right\}. \quad (6.63)$$

Since b_0 as well as L_0^{tot} vanishes on this space, we have $d = \hat{d}$ and hence $\hat{d}^2 = 0$ on \mathcal{F}_0 . Thus one can consider a cohomology with respect to \hat{d} .

Although \hat{d} is simpler than d , it is still quite complicated and one would like to reduce the problem further. This is achieved by decomposing \hat{d} according to the

so called *degree* assigned to appropriate operators. For this purpose, introduce the following light-cone type variables:

$$q^\pm = \frac{1}{\sqrt{2}}(q_1 \pm q_2), \quad p^\pm = \frac{1}{\sqrt{2}}(p_1 \pm p_2), \quad (6.64)$$

$$[q^\pm, p^\pm] = i\hbar, \quad (6.65)$$

$$\alpha_m^\pm = \frac{1}{\sqrt{2}}(\alpha_m^1 \pm \alpha_m^2) \quad [\alpha_m^\pm, \alpha_n^\mp] = m\hbar\delta_{m+n,0}. \quad (6.66)$$

To these operators we assign the degrees

$$\deg(\alpha_n^+) = \deg(c_n) = 1, \quad \deg(\alpha_n^-) = \deg(b_n) = -1, \quad (6.67)$$

$$\deg(\text{Rest}) = 0. \quad (6.68)$$

Then \hat{d} is decomposed according to the degree as

$$\hat{d} = \hat{d}_0 + \hat{d}_1 + \hat{d}_2, \quad (6.69)$$

$$\hat{d}_0 = \sum_{n \neq 0} P^+(n) c_{-n} \alpha_n^-, \quad (6.70)$$

$$\hat{d}_1 = \sum_{n.z.m.} : c_{-n} (\alpha_{-m}^+ \alpha_{m+n}^- + \frac{1}{2}(m-n) c_{-m} b_{m+n} + L^f n) :, \quad (6.71)$$

$$\hat{d}_2 = \sum_{n \neq 0} P^-(n) c_{-n} \alpha_n^+, \quad (6.72)$$

where *n.z.m.* stands for summation over the non-zero modes and

$$P^+(n) = \frac{1}{\sqrt{2}}(p_1 + p_2 + 2iQn), \quad (6.73)$$

$$P^-(n) = p_-. \quad (6.74)$$

Notice that although \hat{d}_1 is still rather involved \hat{d}_0 and \hat{d}_2 have very simple structures. Since they have the lowest and the highest degree in the decomposition (6.69), $\hat{d}^2 = 0$ implies $\hat{d}_0^2 = \hat{d}_2^2 = 0$. Furthermore, it is easy to see that they commute (anti-commute) with L_0^{tot} (b_0). This allows us to consider \hat{d}_0 - or \hat{d}_2 - cohomologies consistently on \mathcal{F}_0 . (Which of these cohomologies is more useful depends on the form of $P^\pm(n)$.) Fortunately these cohomologies are simple enough to be solved explicitly.

Let us give an example. When $P^+(n) \neq 0$ for all non-zero n , one can construct an operator

$$K^+ \equiv \sum_{n \neq 0} \frac{1}{P^+(n)} \alpha_{-n}^+ b_n. \quad (6.75)$$

This operator is kind of an inverse of \hat{d}_0 . Its anti-commutator with \hat{d}_0 yields

$$\begin{aligned} \{\hat{d}_0, K^+\} &= \sum_{n \neq 0} : (nc_{-n} b_n + \alpha_{-n}^+ \alpha_n^-) : + \hbar \\ &= \hat{N}^{dLg}, \end{aligned} \quad (6.76)$$

where \hat{N}^{dLg} is precisely the level-counting operator in the dilaton-Liouville-ghost (d-L-g) sector. This relation, which is quite similar to (6.61), implies that a non-trivial state in \hat{d}_0 cohomology must satisfy

$$\hat{N}^{dLg} | \psi > = 0, \quad (6.77)$$

namely that it should not contain any non-zero mode excitations in the d - L - g sector. Taking this into account, the $L_0^{tot} | \psi > = 0$ condition becomes

$$L_0^{tot} | \psi > = \left(p^+ p^- + \frac{1}{2} \vec{p}_f^2 + \hat{N}^f - \hbar \right) | \psi > = 0. \quad (6.78)$$

This expresses nothing but the energy balance between the matter and the zero modes of dilaton and gravitational fields. In other words, whenever matter is present it gets gravitationally dressed such that the total energy of the system vanishes. The problem has now been reduced to enumerating states satisfying two conditions (6.77) and (6.78) and this can easily be done.

Reconstruction of the relative and the absolute cohomologies from the \hat{d}_0 - (or \hat{d}_2 -) cohomologies is a bit involved. We omit the details and show only the correspondence:

$$\hat{d}_{0,2}\text{-cohomology} \xleftrightarrow{1:1} \hat{d}\text{-cohomology} \xleftrightarrow{1:2} d\text{-cohomology}. \quad (6.79)$$

Just to give you a feeling, let us display the formula which constructs the state $| \psi >$ in the relative cohomology out of a state $| \psi_0 >$ in the \hat{d}_0 -cohomology:

$$| \psi > = \sum_{n=0}^{\infty} (-1)^n (T^+)^n | \psi_0 >, \quad (6.80)$$

$$T^+ \equiv \hat{N}_{dLg}^{-1} K^+ \hat{d}_1. \quad (6.81)$$

Since both K^+ and \hat{d}_1 are rather involved, the expression above appears to be extremely complicated. However, when one starts writing out simple examples, one almost immediately understands what this formula means. Let $| \vec{P} >_{\downarrow}$ be a state of zero energy made out of zero modes p^{\pm} and \vec{p}_f only. Then, for example,

$$\text{For } | \psi_0 > = \alpha_{-1}^i | \vec{P} >_{\downarrow}, \quad (6.82)$$

$$| \psi > = \left[\alpha_{-1}^i - \frac{p_f^i}{P^+(1)} \alpha_{-1}^+ \right] | \vec{P} >_{\downarrow}, \quad (6.83)$$

$$\begin{aligned} \text{For } | \psi_0 > &= \alpha_{-2}^i | \vec{P} >_{\downarrow} \\ | \psi > &= \left[\alpha_{-2}^i - \frac{2}{P^+(1)} \alpha_{-1}^i \alpha_{-1}^+ - \frac{p_f^i}{P^+(2)} \alpha_{-2}^+ \right. \\ &\quad \left. + \frac{p_f^i}{P^+(1)} \left\{ \frac{1}{P^+(1)} + \frac{1}{P^+(2)} \right\} (\alpha_{-1}^+)^2 \right] | \vec{P} >_{\downarrow}. \end{aligned} \quad (6.84)$$

The reader familiar with string theory recognizes that these are essentially the transverse physical states of bosonic string theory. In fact this observation will allow us to find a better representation of physical states in the next section. (Actually, in addition to these “transverse” states, there are so called “discrete states”. However, unlike in the $c \leq 1$ models, they occur only at level 1 and will not play important roles in the subsequent discussions.)

7 Extraction of Space-time Geometry

Thus far, we have been able to quantize the model exactly and obtained all the physical states characterized by the BRST-closed condition $d|\Psi\rangle = 0$. This condition is a quantum expression of the classical constraint $T_{\mu\nu} = 0$ and herein lies the gravitational dynamics. The next task is obviously to try to understand what physics these states describe. But we now face a problem: $|\Psi\rangle$ is an abstract state made up of free field oscillators and due to the general covariance it does not depend on the coordinates. Not a shadow of space-time picture is in sight!

This kind of situation is in fact not uncommon in quantum mechanics. For example, an excited state of a harmonic oscillator of the form $(a^\dagger)^n |0\rangle$ by itself does not tell us anything about its physical content. We must act on it by such physically interpretable operators as the energy, the momentum, and so on and see the response. Or we must compute the expectation values of such operators in these states in order to form a physical picture. So we must do the same for our problem; we must act on $|\Psi\rangle$ by some physics-probing operators in order to get out a space-time picture.

What operators should we use? Preferably, one would like to use BRST (gauge) invariant operators and we do have infinitely many such operators at hand. These are the “vertex operators” familiar in string theory. The problem is that unlike in string theory, where we are interested in the physics in the target space, “vertex operators” are extremely hard to interpret in the context of two-dimensional gravity. Being integrals of local operators, they are coordinate independent and hence their expectation values are just numbers and unfortunately there is yet no scheme of getting physics out of them.

Another suggestion may be to use gauge-invariant “clocks” and “rulers”. While conceptually correct, it is virtually impossible to construct operators that would work like such devices out of the fields in hand.

As a matter of fact, even in classical general relativity, we do not know how to get space-time picture by only using gauge-invariant quantities. What we normally do is to first compute the gauge-variant quantities such as $g_{\mu\nu}$, $R_{\mu\nu}$, R , $T_{\mu\nu}^f$ etc., and then try to extract gauge-invariant consequences. What the principle of general covariance dictates is not that the measurable quantities are observer-independent but that the behavior of quantities measured by different observers is subjected to the same physical laws.

For the reasons just described and in short of a better idea, we shall employ the same procedure as in classical general relativity. Namely, we shall choose a gauge (within the conformal gauge) and compute the mean values $\langle g_{\mu\nu} \rangle$ etc. in some interesting states and interpret them as what we see on the average in these states. In particular, we will try to construct states describing black hole configurations in the limit of large L (the parameter size of the universe) [15], [16].

As we begin putting this idea into practice, we immediately face a couple of problems. First stems from the fact that, in the present context, the Virasoro levels express the discretized energy levels of the fields, not the squared mass as in string theory. If we denote the level by n , we have

$$E_n \propto \frac{n}{L}. \quad (7.1)$$

As we wish to produce configurations where matter fields carry *finite* energy in the limit $L \rightarrow \infty$, we need to control physical states with *arbitrarily high Virasoro levels*. We will shortly describe how it can be done.

Another problem is how we should define the inner product between states, especially for the zero-mode sector, since it is known [53] that hermitian operators with *continuous* spectra need not have real eigenvalues. Thus we must be careful to ensure the reality of physical observables. As this is rather technical, we refer the reader to the original paper[16] and will not elaborate on it further.

7.1 Physical States at Arbitrary Level

Previously, we have obtained a compact formula (6.80) for arbitrary physical states in the BRST formalism. That expression, however, is rather formal and will not be useful for our purposes. Now the fact that these states are essentially the transverse states in string theory suggests a better idea; the use of DDF type spectrum generating operators [54]. Indeed we can prove that the states produced by BRST formalism can be generated by the following oscillators \tilde{A}_m^i :

$$\tilde{A}_m^i \equiv e^{i(m/\tilde{\gamma}p^+)\ln(\tilde{\gamma}p^+)} \int_0^{2\pi} \frac{dy^+}{2\pi} e^{im\eta^+ / (\tilde{\gamma}p^+)} \partial_+ \phi_f^i(y^+), \quad (7.2)$$

where ϕ_f^i is a normalized matter field and it is “dressed” by the field η^+ in the exponent. It is a dimension 0 primary defined by

$$\eta^+ = \ln \left(\exp(\tilde{\gamma}q^+/2) A(x^+) / \mu \right), \quad (7.3)$$

where $A(x^+)$ is the operator defined before, namely

$$A(x^+) = \mu C(\alpha) \int_0^{2\pi} d\sigma' E_\alpha(\sigma - \sigma') e^{\psi_+(\sigma')}. \quad (7.4)$$

(The phase factor in front in (7.2), made up of zero mode p^+ , is there to cancel a phase in the integrand.) It is easy to show that \tilde{A}_m^i 's satisfy the commutation relations for

oscillators, $[\tilde{A}_m^i, \tilde{A}_n^j] = m\delta_{m+n,0}$ and they are BRST invariant. Thus we can build up physical states like $|\psi\rangle = \sum C_{m_1 \dots m_N}^{i_1 \dots i_N} \tilde{A}_{-m_1}^{i_1} \dots \tilde{A}_{-m_N}^{i_N} |\vec{P}\rangle_{\downarrow}$ where $|\vec{P}\rangle_{\downarrow}$ is a BRST invariant zero mode state. An important feature of \tilde{A}_m^i is that it does not contain the field χ . Consequently, ψ is not active upon physical states built this way and it simplifies our calculations.

7.2 Choice of Probing Operators and Macroscopic States

As we have already stated, we shall use gauge-variant but easily interpretable operators to probe physics. Specifically, we will deal with

$$T^f(\xi^+) = \frac{1}{\tilde{\gamma}^2} : (\partial_{\xi^+} \tilde{f})^2 : , \quad (7.5)$$

$$g^{\alpha\beta} = (: \chi e^{-\psi} : + A B e^{-\psi}) \eta^{\alpha\beta} , \quad (7.6)$$

where the normal ordering is defined as usual by

$$: \chi e^{-\psi} : \equiv \chi e^{-\psi} - [\chi_a, e^{-\psi}] , \quad (7.7)$$

with χ_a denoting the annihilation part of χ .

Next, we must fix a gauge within the conformal gauge. Recall that any physical state $|\Psi\rangle$ is of the form

$$|\Psi\rangle = |\Psi_0\rangle + d |\Lambda\rangle , \quad (7.8)$$

$$(7.9)$$

where $|\Psi_0\rangle$ has vanishing weight and constitutes the non-trivial part of the cohomology. We know that a choice of $d |\Lambda\rangle$ part should not change physics (as long as it will not cause anything singular in various quantities). Thus it should correspond to the gauge freedom left in the conformal gauge. Let us give an example. Let \mathcal{O} be a $(1,1)$ operator like the metric $g_{\mu\nu}$. Under the conformal transformation, it transforms as $\delta g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$. For $x^+ \longrightarrow x^+ + \epsilon^+(x^+)$ in conformal gauge, this becomes

$$\delta g_{+-} = \nabla_+ \epsilon_- = \partial_+ \epsilon_- , \quad (7.10)$$

since only non-vanishing component of the Christoffel symbol is Γ_{++}^+ . Now under $|\Lambda\rangle \rightarrow |\Lambda\rangle + |\delta\Lambda\rangle$, the expectation value of \mathcal{O} changes by

$$\begin{aligned} \delta \langle \mathcal{O}(x) \rangle &= \langle \Psi | \mathcal{O}(x) d |\delta\Lambda\rangle + h.c. \\ &= - \langle \Psi | [d, \mathcal{O}(x)] | \delta\Lambda\rangle + h.c. \\ &= - \langle \Psi | \sum_n c_{-n} [L_n, \mathcal{O}(x)] | \delta\Lambda\rangle + h.c. \\ &= \partial_+ \left(- \langle \Psi | c(x^+) \mathcal{O}(x) | \delta\Lambda\rangle + h.c. \right) , \end{aligned} \quad (7.11)$$

and this is precisely of the form (7.10) of a gauge transformation.

What would be a convenient choice for the gauge part $|\Lambda\rangle$? A hint is provided by the following simple observation. Let $|\psi_n\rangle$ be a state with Virasoro weight n , *i.e.* $L_0^{tot}|\psi_n\rangle = n|\psi_n\rangle$. Then for any operator $\mathcal{O}(x^+)$ carrying a global weight, we have

$$\begin{aligned} \langle \psi_n | [L_0^{tot}, \mathcal{O}(x^+)] | \psi_m \rangle &= \frac{1}{i} \partial_+ \langle \psi_n | \mathcal{O}(x^+) | \psi_m \rangle \\ &= (n - m) \langle \psi_n | \mathcal{O}(x^+) | \psi_m \rangle. \end{aligned} \quad (7.12)$$

This gives

$$\langle \psi_n | \mathcal{O}(x^+) | \psi_m \rangle = c_{nm} e^{i(n-m)x^+}, \quad (7.13)$$

where c_{nm} is a constant. This implies that (i) $\langle \Psi_0 | \mathcal{O}(x^+) | \Psi_0 \rangle$ can only be a constant and (ii) if we wish to produce various coordinate dependence, we should take the gauge part to be a superposition of states with various Virasoro levels.

With this in mind, we shall take our physical state to be of the form (now including both left and right moving modes)

$$|\Psi\rangle = |\Psi_0\rangle + \frac{1}{\kappa} (d b_{-M} + \bar{d} \bar{b}_{-M}) |\Omega\rangle, \quad (7.14)$$

where b_{-M} (\bar{b}_{-M}) is the left (right) going anti-ghost at level M and the constant κ will be taken to be of order \hbar^2 . The state $|\Omega\rangle$, which constitutes the gauge part, is made up only of zero-modes (zero-modes do carry Virasoro weights), and is give by

$$|\Omega\rangle \equiv \sum_{k=-\infty}^{\infty} \omega_k \sum_{l=\pm 1,0} |\tilde{P}(k, l)\rangle, \quad (7.15)$$

$$\begin{aligned} |\tilde{P}(k, l)\rangle &\equiv e^{-icp^+/\hbar\gamma\gamma^2} \int_{-\infty}^{\infty} dp^+ \int_{-\infty}^{\infty} dp^- W(p^+) \\ &\quad \cdot \delta(p^- - \frac{1}{p^+}(\hbar - \frac{1}{2}p_f^2)) |p^+, p^-(k, l), \vec{p}_f\rangle, \end{aligned} \quad (7.16)$$

$$p^-(k, l) \equiv p^- - \frac{\hbar}{p^+} k - i\hbar\tilde{\gamma}l. \quad (7.17)$$

It may look a bit complicated but it satisfies minimum requirements for a good gauge part. Let us briefly explain various factors: ω_k is a constant, the exponential factor in front of the integral is BRST invariant and will be used to cancel certain unwanted terms in the mean value $\langle g^{\mu\nu} \rangle$, $W(p^+)$ is a smearing function to make certain integrals finite, the δ -function enforces the zero-energy condition for the zero mode state with $(k, l) = (0, 0)$, and the two types of shifts from p^- in the definition of $p^-(k, l)$ are needed to make certain mean values non-trivial and real. ω_k and $W(p^+)$ will be given explicitly later.

Now as for the non-trivial part $|\Psi_0\rangle$, we want it to be able to describe states where matter shock wave with finite energy is present and produces a black hole. Such a macroscopic configuration should be naturally described by a coherent state. Thus, by using the physical oscillators \tilde{A}_n (for simplicity, we consider cases where only one of the matter fields is excited and hence drop the superscript), we define

$$|\Psi_0\rangle \equiv e^G |\tilde{P}\rangle, \quad G \equiv \frac{1}{\hbar} \sum_{n \geq 1} \frac{\tilde{\nu}_n}{n} \tilde{A}_{-n}, \quad (7.18)$$

$$\tilde{\nu}_n = \nu_n e^{inx_0^+} \quad (\nu_n, x_0^+ : \text{real constants}), \quad (7.19)$$

$$|\tilde{P}\rangle \equiv |\tilde{P}(0,0)\rangle. \quad (7.20)$$

By construction $|\Psi_0\rangle$ is indeed a coherent state and satisfies $\tilde{A}_n |\Psi_0\rangle = \tilde{\nu}_n |\Psi_0\rangle$. x_0^+ will turn out to specify the location of the shock wave.

7.3 Emergence of Black Hole Geometry

Having prepared the physical states, we can start computing the average values of the physics-probing operators of our choice $g^{\mu\nu}, T^f(x^+)$ in such states. Computations are long but can be carried out exactly. The results, with the parameters such as ω_k, ν_n etc. left unspecified, still contain infinite sums and look more or less like a mess. We shall not display them. However, as we choose our parameters appropriately and take the large L limit, drastic simplification occurs and one begins to see a picture of the averaged space-time. The reason for this simplification is that the terms expressing contributions from modes at finite Virasoro levels all die away in this limit and furthermore the infinite sums can be replaced by tractable integrals.

Let us now describe what we get. First we must specify the parameters concretely. Since the energy variable $u \equiv n/L$ tends to be continuous as $L \rightarrow \infty$, dependence on the level n should be replaced by that on u . With this in mind we choose the parameters as

$$\nu_n = \nu(Lu) = \nu u^d e^{-au^2}, \quad (7.21)$$

$$\omega_n = \omega(Lu) = \frac{-\omega}{Lu} \quad (n \neq 0), \quad (7.22)$$

$$\omega = \text{a positive constant}, \quad (7.23)$$

$$\omega_0 = \text{a constant to be adjusted}, \quad (7.24)$$

$$W(p^+) = p^+ e^{-\alpha(p^+ - p_0^+)^2/2}. \quad (7.25)$$

In the expression for ν_n , d and a are constants and they control, respectively, the type of matter distribution and the width of its flux.

The most typical configuration turned out to be produced by the choice $d = -1/2$. In this case, the average behavior of the left-going matter energy-momentum tensor and that of the inverse metric are given by

$$\langle T^f(\xi^+) \rangle \stackrel{L \rightarrow \infty}{\sim} \omega \nu^2 I_T(\xi^+ - \xi_0^+) \langle \tilde{P} | \tilde{P} \rangle, \quad (7.26)$$

$$\begin{aligned} \langle g^{-1} \rangle &\equiv \langle : (\chi + AB) e^{-\psi} : \rangle \\ &\stackrel{L \rightarrow \infty}{\sim} \left(K_g(\xi^+) + \lambda_g^2 \xi^+ \xi^- \right) \langle \tilde{P} | \tilde{P} \rangle . \end{aligned} \quad (7.27)$$

In the expressions above, λ_g^2 is a constant proportional to λ^2 , the common factor $\langle \tilde{P} | \tilde{P} \rangle$ is the norm of the zero-mode state, which can be made finite, the function $K_g(\xi^+)$ is given by

$$K_g(\xi^+) = c_K \xi^+ + d_K - \tilde{\gamma}^2 \omega \nu^2 I_\chi(\xi^+ - \xi_0^+) \quad (c_K, d_K : \text{constants}), \quad (7.28)$$

and $I_T(\xi)$ and $I_\chi(\xi)$ are integrals of the form

$$I_\chi(\xi) = \int_{1/L}^{\infty} du \frac{\cos u \xi}{u^2} \int_0^u dv [v(u-v)]^{-1/2} e^{-a(v^2+(u-v)^2)}, \quad (7.29)$$

$$I_T(\xi) = \int_{1/L}^{\infty} du \cos u \xi \int_0^u dv [v(u-v)]^{-1/2} e^{-a(v^2+(u-v)^2)}. \quad (7.30)$$

It is easy to see that these integrals are related by

$$I_T(\xi) = -\partial_\xi^2 I_\chi(\xi). \quad (7.31)$$

This actually expresses the fundamental energy balance condition and is a good check on our calculations. To see this, recall the energy-momentum constraint:

$$\tilde{\gamma}^2 T_{++} = \partial_+ \chi \partial_+ \psi - \partial_+^2 \chi + \frac{1}{2} (\partial_+ f)^2 = 0. \quad (7.32)$$

As we remarked before, on the physical states made up of \tilde{A}_n the field ψ is inactive and can be regarded to vanish. Thus the constraint takes the form

$$\frac{1}{2} (\partial_+ f)^2 = \partial_+^2 \chi, \quad (7.33)$$

which is nothing but the relation between the integrals above.

As for the evaluation of these integrals, one can perform the oscillatory integral over u analytically and then do the remaining integral by numerical methods. The results are plotted in Fig.8a and 8b.

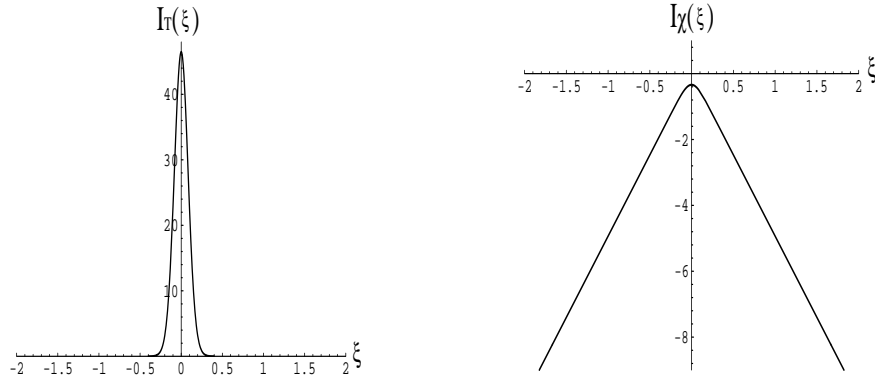


Fig.8a Plot of the integral $I_T(\xi)$. **Fig.8b** Plot of the integral $I_\chi(\xi)$.

$I_T(\xi)$ is very close to a Gaussian, width of which is controled by the parameter a and for small a it approaches a δ function and describes a matter shock wave. As for $I_\chi(\xi)$ (up to a constant $L\pi$, which can be canceled by a constant d_K), it behaves roughly like $-(\pi^2)|\xi|$ except near the origin and as $a \rightarrow 0$ it converges to that function. Remarkably, these functions behave exactly like the corresponding ones in the classical CGHS model in the limit of small a !

One then expects that a black hole is generated due to the matter flux. To confirm this, let us compute the curvature scalar. It is given by

$$R^g = 4\lambda_g^2 \frac{K_g(\xi^+) - \xi^+ \partial_+ K_g(\xi^+)}{K_g(\xi^+) + \lambda_g^2 \xi^+ \xi^-}. \quad (7.34)$$

With appropriate choices of parameters, K_g takes the form

$$K_g(\xi^+) = \tilde{\gamma}^2 \omega \nu^2 \left(\frac{\pi^2}{2} (\xi^+ - \xi_0^+) - I_\chi(\xi^+ - \xi_0^+) + L\pi \right), \quad (7.35)$$

which for $a \ll (\xi^+ - \xi_0^+)^2$ goes like

$$K_g(\xi^+) \sim (\pi^2/2) \tilde{\gamma}^2 \omega \nu^2 (\xi^+ - \xi_0^+) \theta(\xi^+ - \xi_0^+). \quad (7.36)$$

The line of curvature singularity resulting from this expression is plotted in Fig.9.

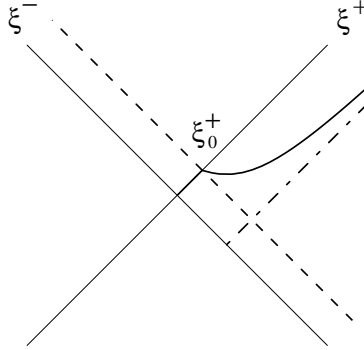


Fig.9 Curvature singularity for $d = -1/2$.

It clearly describes the formation of a black hole by a matter shock wave just as in the CGHS model.

If we change the value of the parameter d controlling the distribution of the matter field, we can get different configurations. For example, if we take $d = 1/2$, it turned out that we get a space-time with a naked singularity depicted in Fig.10.

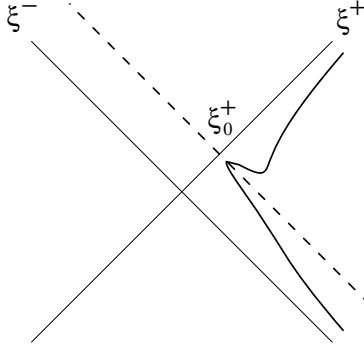


Fig.10 Curvature singularity for $d = 1/2$.

We would like to emphasize that although the space-time pictures that emerged in the examples above look “classical” actually all the quantum corrections have been fully taken into account. In fact the very notion of “quantum correction” becomes ambiguous in exact treatment since we do not have a classical background to begin with. As is apparent, the coherent state $|\Psi_0\rangle$ is a highly non-perturbative quantum state and precisely through its *quantum coherence* macroscopic classical-like configurations are formed.

8 Future Problems

We hope to have convinced the reader that our attempt, despite the use of a simple model with a number of shortcomings, has produced certain success. Namely, we have shown that a certain model of CGHS type can be quantized exactly and further we have explicitly implemented an idea of getting physical pictures out of abstract physical states. On the other hand, our analysis also illustrated the nature of difficulties that one must face in any challenge for exact treatment of quantum gravity. We now list some of the important problems left unsolved.

The most serious among them is *how to define and compute the S -matrix*, which is a prerequisite for proper analysis of the Hawking radiation. First, the definition of scattering matrix calls for a separation of bulk geometry and particle excitations. In the semi-classical treatment, this is not a serious problem since the space-time picture (*i.e.* the background geometry) is already available *before* one starts discussing the S -matrix. On the other hand, in an exact treatment, “geometry” emerges only *after* the computation of some expectation values. This is a universal difficulty and not singular to our particular treatment since by the very requirement of general covariance the physical states by themselves cannot contain any dependence on the coordinates.

Deeply related to the point above is the fact that the definition of the S -matrix requires a separation of sub-systems or sub-regions. For instance, the S -matrix is supposed to relate the states in the “far past” to those in the “far future” but the notion of “time” itself is, to say the least, hard to come by.

Another famous problem one must face in an attempt for exact quantization is the question of quantum (in)coherence. Logically, pure states cannot possibly evolve into mixed states, as is the case in our treatment. This would mean that the apparent quantum incoherence implied by the thermal spectrum of Hawking radiation can only be due to the semi-classical approximation used in deriving it. As this is a general problem in the quantum mechanics of macroscopic objects in interaction with microscopic states, it would be interesting to construct a simple model to demonstrate this.

Finally, let us mention the problem of how to define, at least conceptually, the notion of “quantum states of a black hole”. This is again a very hard problem in the exact treatment since at present one has no idea how to characterize a quantum state which contains a black hole. One would have to somehow bring in the notion of causal structure at the abstract level. Any such idea would be extremely interesting since, as is well known, this problem is directly related to the question of the statistical meaning of the black hole entropy and of the no hair theorem.

We hope that some of these problems can be solved, at least in a clever model, in the near future.

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References

- [1] S.W. Hawking, Commun. Math. Phys. **43**, 199 (1975); Phys. Rev. **D14**, 2460 (1976).
- [2] S.W. Hawking, Commun. Math. Phys. **87**, 395 (1982).
- [3] J.D. Bekenstein, Nuovo Cim. Lett. **4**, 737 (1972); Phys. Rev. **D7**, 2333 (1973).
- [4] S.W. Hawking, Monthly Notices Roy. Astron. Soc. **152**, 75 (1971); A. Salam in *Quantum Gravity: an Oxford Symposium* (Eds. Isham, Penrose and Sciama, Oxford Univ. Press, 1975); G. 't Hooft, Nucl. Phys. **B335**, 138 (1990).
- [5] E.F.E. Holzhey and F. Wilczek, Nucl. Phys. **B380**, 447 (1992).
- [6] M.J. Duff, R.R. Khuri, R. Minasian and J. Rahmfeld, Nucl. Phys. **B418**, 195 (1994); M.J. Duff and J. Rahmfeld, "Massive String States as Extreme Black Holes", Texas A and M University Report CTP-TAMU-25/94, hep-th/9406105.
- [7] R.M. Wald, *General Relativity*, The Univ. of Chicago Press, Chicago, (1984).
- [8] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, W.H. Freeman and Company, (1973).
- [9] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-time*, Cambridge Univ. Press, (1973).
- [10] C.G. Callan, S.B. Giddings, J.A. Harvey and A. Strominger, Phys. Rev. **D45**, R1005 (1992).
- [11] K. Baradakei, M. Crescimanno and E. Rabinovici, Nucl. Phys. **B344**, 344 (1990).
- [12] E. Witten, Phys. Rev. **D44**, 314 (1991).
- [13] G. Mandal, A. Sengupta, and W. Wadia, Mod. Phys. Lett. **A6**, 1685 (1991).
- [14] S. Hirano, Y. Kazama and Y. Satoh, Phys. Rev. **D48**, 1687 (1993).
- [15] Y. Kazama and Y. Satoh, Phys. Rev. **D50**, R2368 (1994).
- [16] Y. Kazama and Y. Satoh, Phys. Rev. **D50**, 3889 (1994).
- [17] E. Verlinde and H. Verlinde, Nucl. Phys. **B406**, 43 (1993); K. Schoutens, E. Verlinde and H. Verlinde, Phys. Rev. **D48**, 2690 (1993).
- [18] J.A. Harvey and A. Strominger, Chicago University Report No. EFI-92-41 (unpublished).
- [19] S.B. Giddings, University of California (Santa Barbara) Report No. UCSBTH-92-36, Erice Theor. Phys. 437 (1992).
- [20] G. Gibbons and K. Maeda, Nucl. Phys. **B298**, 741 (1988).
- [21] D. Garfinkle, G. Horowitz, and A. Strominger, Phys. Rev. **D43**, 3140 (1991); Erratum: *ibid* **D45**, 3888 (1992).
- [22] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B371**, 269 (1992).
- [23] A. Jevicki and T. Yoneya, Nucl. Phys. **B411**, 64 (1994).
- [24] S.M. Christensen and S.A. Fulling, Phys. Rev. **D15**, 2088 (1977).

- [25] S.B. Giddings and W.M. Nelson, Phys. Rev. **D46**, 2486 (1992).
- [26] J.G. Russo, L. Susskind and L. Thorlacius, Phys. Lett. **B292**, 13 (1992).
- [27] T. Banks, A. Dabholkar, M. Douglas and M. O’Loughlin, Phys. Rev. **D45**, 3607 (1992).
- [28] L. Susskind, L. Thorlacius, Nucl. Phys. **B382**, 123 (1992).
- [29] J. Russo, L. Susskind and L. Thorlacius, Phys. Rev. **D46**, 3444 (1992).
- [30] S. Hawking, Phys. Rev. Lett. **69**, 406 (1992).
- [31] B. Birnir, S.B. Giddings, J. Harvey and A. Strominger, Phys. Rev. **D46**, 638 (1992).
- [32] A. Strominger, Phys. Rev. **D46**, 4396 (1992).
- [33] A. Miković, Phys. Lett. **B291**, 19 (1992); *ibid* **B304**, 70 (1993).
- [34] D. Lowe, Phys. Rev. **D47**, 2446 (1993).
- [35] S. W. Hawking and J. M. Stewart, Nucl. Phys. **B400**, 393 (1993).
- [36] T. Piran and A. Strominger, Phys. Rev. **D48**, 4729 (1993).
- [37] L. Susskind, L. Thorlacius and J. Uglum, Phys. Rev. **D48**, 3743 (1993).
- [38] S. W. Hawking and Hayward, Phys. Rev. **D49**, 5252 (1994).
- [39] S.P. de Alwis, Phys. Lett. **B289**, 278 (1992); Phys. Rev. **D46**, 5429 (1992); Phys. Lett. **B300**, 330 (1993).
- [40] S.B. Giddings and A. Strominger, Phys. Rev. **D47**, 2454 (1993).
- [41] A. Bilal and C.G. Callan, Nucl. Phys. **B394**, 73 (1993).
- [42] J. Russo, L. Susskind and L. Thorlacius, Phys. Rev. **D46**, 3444 (1992).
- [43] Y. Kazama and Y. Satoh, “A Unified Approach to Solvable Models of Dilation Gravity in Two-Dimensions Based on Symmetry”, University of Tokyo, Komaba, Report No.UT-Komaba 94-16, hep-th @xxx/9409179.
- [44] J.G. Russo and A. Tseytlin, Nucl. Phys. **B382**, 259 (1992).
- [45] K. Hamada and A. Tsuchiya, Int. J. Mod. Phys. **A8**, 4897 (1993).
- [46] F. David, Mod. Phys. Lett. **A3**, 1651 (1988);
J. Distler and H. Kawai, Nucl. Phys. **B321**, 509 (1989).
- [47] H.J. Otto and G. Weigt, Phys. Lett. **B159**, 341 (1985).
- [48] Y. Kazama and H. Nicolai, Int. J. Mod. Phys. **A9**, 667 (1994).
- [49] B.H. Lian and G.J. Zuckerman, Phys. Lett. **B266**, 21 (1991); Phys. Lett. **B254**, 417 (1991); Comm. Math. Phys. **145**, 561 (1992).
- [50] P. Bouwknegt, J. McCarthy, and K. Pilch, Comm. Math. Phys. **145**, 541 (1992).
- [51] A. Bilal, Phys. Lett. **B282**, 309 (1992).
- [52] Y. Matsumura, N. Sakai, Y. Tanii, and T. Uchino, Mod. Phys. Lett. **A8**, 1507 (1993).

- [53] H. Arisue, T. Fujiwara, T. Inoue, and K. Ogawa , J. Math. Phys. **9**, 2055 (1981); H. Arisue, T. Fujiwara, M. kato, and K. Ogawa , Phys. Rev. **D35** , 2309 (1987).
- [54] E. Del Giudice, P. Di Vecchia and S. Fubini, Ann. Phys. **70**, 378 (1972).